Nonlinearity 17 (2004) 23–47 PII: S0951-7715(04)52658-0

Fronts in extended systems of bistable maps coupled via convolutions

Ricardo Coutinho[1] **and Bastien Fernandez**[2]

[1] Departamento de Matemática, Instituto Superior Técnico, Av. Rovisco Pais 1096, Lisboa Codex, Portugal

[2] Centre de Physique Théorique (FRUMAM), CNRS Luminy Case 907, 13288 Marseille Cedex 09, France

E-mail: rcoutin@math.ist.utl.pt and bastien@cpt.univ-mrs.fr

Received 19 August 2002, in final form 29 August 2003
Published 26 September 2003
Online at stacks.iop.org/Non/17/23

Recommended by L Bunimovich

Abstract

An analysis of front dynamics in discrete time and spatially extended systems with general bistable nonlinearity is presented. The spatial coupling is given by the convolution with distribution functions. It allows us to treat in a unified way discrete, continuous or partly discrete and partly continuous diffusive interactions. We prove the existence of fronts and the uniqueness of their velocity. We also prove that the front velocity depends continuously on the parameters of the system. Finally, we show that every initial configuration that is an interface between the stable phases propagates asymptotically with the front velocity.

Mathematics Subject Classification: 37L99, 39B22, 46T20

1. Systems of bistable maps coupled by convolutions

Fronts between two stable phases is a widespread phenomenon in spatially extended systems. Such nonlinear waves are believed to emerge in the presence of a bistable nonlinearity and a diffusive coupling. In particular, they should manifest themselves independently of the discrete or continuous nature of time in the system, and independently of the discrete or continuous nature of the spatial coupling. From the mathematical point of view, the existence of fronts and related properties were analysed separately in continuous time and in discrete time systems. Most studies have considered the continuous case.

In continuous time models (i.e. differential equations), the first results on the existence of fronts were obtained for the Fisher–Kolmogorov PDE (see [9] and references therein). The results were extended to other continuous time systems with continuous spatial coupling [3]. In particular, we mention the case of integro-differential equations with interaction given by the convolution with an absolutely continuous function [1]. These questions were also addressed in models with discrete coupling (i.e. in lattices of coupled ODEs). The existence of a pinning effect, namely the structural stability of fronts with zero velocity, was one of the first mathematical results obtained for such systems [11]. Recently, the existence of fronts and the uniqueness of their velocity were proved in coupled ODEs with finite-range nonlinear coupling [14].

In discrete time systems, the existence of fronts and the uniqueness of their velocity were proved in the case where the bistable nonlinearity is piecewise affine [4, 7]. For discrete couplings (i.e. for coupled map lattices [10]), mode-locking of the front velocity in the parameter space was proved. In particular, a pinning effect was obtained that corresponds to the plateau of zero velocity. Despite these results for piecewise affine systems, an analysis of front dynamics in discrete time systems with general nonlinearity and general coupling has not been done, to the best of our knowledge.

In this paper, we consider discrete time systems with arbitrary nonlinearity. The couplings are given by the convolution with distribution functions as introduced in [7]. In a first step, we present a detailed analysis for systems based on a unique nonlinearity and a unique distribution function (sections 2–4). In a second step (section 5), the results are extended to systems that are convex linear combinations of systems of the previous type. Since the arguments are similar to those in sections 2–4, section 5 is less formal and we only present sketches of the proofs.

For all these systems, we prove the existence of fronts and the uniqueness of their velocity (theorem 1.1 and statement 1 of theorem 5.1). Contrary to the case of continuous time systems, the uniqueness of front shape does not hold generally. As a consequence, Lyapunov stability of fronts may only be shown locally in phase space. Instead of considering this property, we define a weaker property that we prove to be global in the set of initial conditions with interfacial profile: the existence and uniqueness of the velocities of subsequent orbits (theorem 1.3 and statement 3 in theorem 5.1).

The choice of convolution couplings allows us to include, in a unified framework, systems with discrete couplings and those with continuous coupling. The first are obtained when choosing a lattice distribution function. The system then reduces to a coupled map lattice. The second are discrete time analogues of continuous time systems with continuous integral couplings such as in [8]. In addition, convolution couplings allow one to represent systems where the interaction is partly discrete and partly continuous. Such interactions are obtained for distribution functions that are a convex combination of a discrete distribution function and a continuous distribution function.

Furthermore, by enlarging gradually the set of points involved in the interaction, one can study front dynamics changes when the coupling varies from a discrete to a continuous one. This is achieved by considering a sequence of distribution functions converging in Hausdorff topology. In particular, theorem 1.2 below implies that the front velocity changes continuously in this case. Stated otherwise, sufficiently small errors in the choice of the interaction result in arbitrarily small errors in the front velocity.

Finally, the present analysis gives results for the planar front dynamics in multidimensional coupled map lattices. Multidimensional coupled map lattices are discrete time systems on configurations over $\mathbb{Z}^d$. Their dynamics is generated by the composition of a bistable interval map and a diffusive coupling [10]. As shown in section 5.3 of [7], the dynamics of planar fronts is governed by a system of bistable maps coupled by convolutions. The corresponding distribution function is a step function whose discontinuities depend on the planar front direction. By applying theorem 1.1 to such systems, the existence of planar fronts and the uniqueness of their velocity are deduced. Moreover, theorem 1.2 implies that the planar front velocity depends continuously on the front direction.

1.1. Definitions

The phase space is the set $\mathcal{B}$ of Borel-measurable functions defined on $\mathbb{R}$ with values in $[0, 1]$. The notation $\|\cdot\|$ always means the supremum norm, and applies either to elements in $\mathcal{B}$ or to functions defined on finite intervals.

Diffusive couplings. In order to introduce convolutions, we recall that a *distribution function*, say h, is a right continuous increasing function defined on $\mathbb{R}$ such that $h(-\infty) = 0$ and $h(+\infty) = 1$. The simplest distribution function is the Heaviside function, H.

$$H(x) = \begin{cases} 0 & \text{if } x < 0, \\ 1 & \text{if } 0 \leqslant x. \end{cases}$$

The *translation* by $v \in \mathbb{R}$ is the operator acting on $\mathcal{B}$ defined by $T^v u(x) = u(x - v)$ for all $x \in \mathbb{R}$. A distribution function h is said to be *degenerate* if $h = T^v H$ for some $v \in \mathbb{R}$.

Given $u \in \mathcal{B}$, the *convolution* $h * u$ with the distribution function h is (well-)defined by the Lebesgue–Stieltjes integral,

$$h * u(x) = \int_{\mathbb{R}} u(x - y)\,\mathrm{d}h(y), \qquad x \in \mathbb{R}$$

and keeps $\mathcal{B}$ invariant.

The convolution with h, viewed as an operator acting on $\mathcal{B}$, is linear and continuous. More precisely, $\|h * u\| \leqslant \|u\|$ for every $u \in \mathcal{B}$. Furthermore, it has three basic properties [7].

- The first property is *positivity*: if u is a non-negative function, then $h * u$ is also non-negative.
- The second property is *homogeneity*, namely commutation with translations: $h * T^v u = T^v h * u$ for every $u \in \mathcal{B}$ and $v \in \mathbb{R}$.

  In order to state the third property, we introduce the symbol $\lim_{n\to\infty} u_n = u$, which denotes the *pointwise convergence* of the sequence of functions $\{u_n\}$. It means that $\lim_{n\to\infty} u_n(x) = u(x)$ for all $x \in \mathbb{R}$ if one deals with functions in $\mathcal{B}$ or for all $x \in [0, 1]$ if one deals with functions of the interval.
- The third property is *s-homogeneity*. An operator on $\mathcal{B}$ is s-homogeneous iff it is homogeneous and commutes with the pointwise limit of sequences in $\mathcal{B}$: if $u_n \in \mathcal{B}$ is such that $\lim_{n\to\infty} u_n = u$, then $\lim_{n\to\infty} h * u_n = h * u$.

  The condition of being s-homogeneous is necessary and sufficient for a bounded linear operator acting on Borel-measurable functions to be a convolution with a distribution function (see [7]).

We shall often use the *projection* P_ℓ (respectively P_r) on left continuous (respectively right continuous) functions. This operator is defined for every increasing function u as follows:

$$P_\ell u(x) = \lim_{\substack{y\to x\\ y<x}} u(y) \quad (\text{respectively } P_r u(x) = \lim_{\substack{y\to x\\ y>x}} u(y)), \qquad x \in \mathbb{R}.$$

The s-homogeneity implies that every convolution commutes with P_ℓ (respectively P_r).

Finally, given a distribution function h, we denote by h^{*n} the *n-fold convolution* defined by

$$h^{*0} = H \qquad \text{and} \qquad h^{*(n+1)} = h^{*n} * h, \quad n \in \mathbb{N}.$$

We shall need other properties of convolutions as well as results on the convergence of sequences of increasing functions. All these results are stated and proved in appendices A and B.

As mentioned in the introduction, the advantage of dealing with convolutions is that the structure of the underlying space where the diffusion acts depends on h, specifically on its *support*, i.e. the set of growth points of h (see examples later).

Bistable maps. The coupling being defined, we introduce the local map which plays the role of a local force. This map is a *bistable map*, that is to say a continuous increasing map $f : [0, 1] \to [0, 1]$ such that there exists $c \in (0, 1)$ so that $f(x) < x$ for all $x \in (0, c)$ and $x < f(x)$ for all $x \in (c, 1)$.

This definition implies that the points 0 and 1 are stable fixed points and that c is an unstable fixed point.

A bistable map is said to be *regular* if there exists $\delta > 0$ so that $|f(x) - f(y)| \leqslant |x - y|$ for all $x, y \in (0, \delta)$ and for all $x, y \in (1 - \delta, 1)$.

The dynamical system. Given a distribution function h and a bistable map f (whose unstable fixed point is always denoted by c in what follows), we consider the dynamical system $(\mathcal{B}, F)$ where F is defined by

$$Fu = h * f \circ u, \qquad u \in \mathcal{B}$$

and is denoted by $h * f$. This map F is well-defined and keeps $\mathcal{B}$ invariant. So, the orbits $\{u^t\}_{t\in\mathbb{N}}$ where $u^{t+1} = Fu^t$ and $u^0 \in \mathcal{B}$ are well-defined.

The main properties of the dynamics are the following. Consider the following usual partial order in $\mathcal{B}$: given two functions $u, u' \in \mathcal{B}$, we say that $u \leqslant u'$ if $u(x) \leqslant u'(x)$ for all $x \in \mathbb{R}$. Positivity of the convolution with h and monotony of f imply monotony of F; namely, if $u \leqslant u'$, then $Fu \leqslant Fu'$. Moreover, s-homogeneity of the convolution and continuity of f imply s-homogeneity of F. In particular, F commutes with both P_ℓ and P_r.

Examples: Coupled map lattices. Consider a lattice distribution function, that is to say, the distribution function h, defined by

$$h(x) = \sum_{\substack{n\in\mathbb{Z}\\ n\leqslant x}} \ell_n,$$

where all $\ell_n \geqslant 0$ and $\sum_{n\in\mathbb{Z}} \ell_n = 1$. In this case, the map F is $Fu(x) = \sum_{n\in\mathbb{Z}} \ell_n f \circ u(x - n)$ and we actually have a dynamics on the lattice $\mathbb{Z}$. This model is called a coupled map lattice [10].

Integral formulation of the classical diffusion. Consider the absolutely continuous distribution function with the heat kernel

$$h(x) = \int_{-\infty}^{x} \mathrm{e}^{-\pi y^2}\,\mathrm{d}y, \qquad x \in \mathbb{R}.$$

The map $Fu(x) = (h * f \circ u)(x) = \int_{\mathbb{R}} f \circ u(x - y)\mathrm{e}^{-\pi y^2}\,\mathrm{d}y$ gives an integral formulation of a reaction–diffusion process in discrete time.

1.2. Results on the dynamics of fronts

We are interested in particular orbits of $(\mathcal{B}, F)$, namely fronts. A front is a travelling wave whose shape is an interface between the stable points 0 and 1. Specifically, a *front of velocity* v is an orbit $\{u^t\}_{t\in\mathbb{N}}$ such that

$$u^t(x) = \phi(x - vt), \qquad x \in \mathbb{R}, \quad t \in \mathbb{N},$$

where the shape ϕ is a distribution function. There exist fronts of velocity v iff there exists a distribution function ϕ that is a solution of the front equation $F\phi = T^v\phi$. Note that the right continuity of ϕ is arbitrary. Indeed, by s-homogeneity, the existence of fronts with left continuous shape is equivalent to the existence of fronts with right continuous shape.

Proceeding to the analysis of the front equation, we obtain the following results on the existence of fronts and the uniqueness of their velocity.

Theorem 1.1. *For any distribution function h and any bistable map f, the map $F = h * f$ has fronts of velocity v for some $v \in \mathbb{R}$. In addition, if f is regular, then this velocity is unique.*

Assume that f is regular and let $v(f, h)$ be the unique front velocity of $h * f$. The next theorem tells us that this velocity depends continuously on the coupling and on the local map. In order to make this statement, we need the following distance in the set of distribution functions (which is equivalent to the Hausdorff distance restricted to graphs of such functions) [7]. Given two distribution functions h and h', let

$$d(h, h') = \inf\{\varepsilon > 0 \ : \ h(x - \varepsilon) - \varepsilon \leqslant h'(x) \leqslant h(x + \varepsilon) + \varepsilon, \ \forall x \in \mathbb{R}\}.$$

The convergence with respect to this distance coincides with the usual convergence of distribution functions (see lemma B.3).

Theorem 1.2. *Let $\{f_n\}_{n\in\mathbb{N}}$ be a sequence of regular bistable maps that converges pointwise to a bistable regular map f. Let $\{h_n\}_{n\in\mathbb{N}}$ be a sequence of distribution functions and h be a distribution function such that* $\lim_{n\to\infty} d(h_n, h) = 0$. *Then* $\lim_{n\to\infty} v(f_n, h_n) = v(f, h)$.

Moreover, not only does the front velocity depend continuously on the local map with 0 and 1 as stable fixed points, but it also depends continuously on the location of these fixed points. Indeed, in theorem 1.2, one can assume that the stable fixed points of f_n are a_n and b_n. (By a linear change of variable, one can extend theorem 1.1 to maps F for which the stable fixed points of f are any real numbers a and b.) Then, for the pointwise convergence to be meaningful, one defines the map f and all the f_n on the interval $[\inf_{n\in\mathbb{N}} a_n, \sup_{n\in\mathbb{N}} b_n]$ by extending these functions, when necessary, to a constant function outside their original definition interval.

Once the existence of fronts has been established, their asymptotic stability can be analysed. In continuous time systems with continuous couplings, fronts were proved to be globally stable [3, 9]. In such systems, their shape is unique up to translations and every orbit with interfacial initial condition approaches asymptotically such a travelling wave.

In discrete time systems, the global stability of fronts may hold, but is not a generic property. Indeed, in some systems, for a large set of parameters, the stability of fronts was shown to hold only locally in phase space [6]. One reason is the existence of several front shapes (not identifiable by applying translations). Another reason is the existence of quasi-fronts. Quasi-fronts are orbits obtained by iterating distribution functions that are solutions of the equation $F^n\phi = T^{nv}\phi$ for some $n > 1$ but not for $n = 1$. When quasi-fronts exist, fronts are not globally stable.

In the short term, in discrete time systems, depending on the parameters, the asymptotic stability of fronts may be global or only local. Nevertheless, one can prove a kind of stability that is global and valid in all cases, namely the existence and the uniqueness of the velocities of interfacial orbits.

Recall that c denotes the unstable fixed point of f. An *interface* is a function $u \in \mathcal{B}$ such that there exist $c_- \in (0, c)$, $c_+ \in (c, 1)$ and $j_1 \leqslant j_2 \in \mathbb{R}$ so that $u(x) \leqslant c_-$ if $x \leqslant j_1$ and $u(x) \geqslant c_+$ if $x \geqslant j_2$. Note that, in part of the literature (see, e.g., [4, 5]), the term interface has a slightly different meaning and denotes the region of transition between the stable phases, namely the interval $[j_1, j_2]$ in our definition.

Given $u \in \mathcal{B}$ and a number $a \in (0, 1)$, consider the quantity (which may be infinite)

$$J_a(u) = \inf\{x \in \mathbb{R} : u(x) \geqslant a\}.$$

One can show that the image under F of every interface is an interface and that for any $a \in (0, 1)$, the quantity $J_a(F^t u)$ is a finite number for sufficiently large t (see beginning of section 4.4). Our last main result states that even though an interfacial orbit may not approach a front, it has asymptotically the front velocity.

Theorem 1.3. *Let h be a distribution function and let f be a regular bistable map. For every interface u and every $a \in (0, 1)$, we have*

$$\lim_{t\to\infty} \frac{J_a(F^t u)}{t} = v(f, h).$$

As stated in the introduction, the results of this section are extended to more general models in section 5.

2. Sub-fronts and their properties

2.1. Definitions

Let $\mathcal{I} \subset \mathcal{B}$ be the subset of increasing functions. The set of *sub-fronts* of velocity v is defined as follows: given $v \in \mathbb{R}$ and $c_+ \in (c, 1)$, let $\mathcal{S}_{v,c_+}$ be defined as

$$\mathcal{S}_{v,c_+} = \{\psi \in \mathcal{I} : F\psi \leqslant T^v\psi \text{ and } J_{c_+}(\psi) = 0\}.$$

A first result indicates that this definition is meaningful.

Lemma 2.1.

(1) There exists $v \in \mathbb{R}$ such that, for every $c_+ \in (c, 1)$, the set $\mathcal{S}_{v,c_+}$ is not empty.
(2) For every $\psi \in \mathcal{S}_{v,c_+}$, we have $\psi(-\infty) \leqslant c$ and $\psi(+\infty) = 1$.

Proof. (1) Given $c_- \in (0, c)$, let the function ψ_{c_-} be defined as $\psi_{c_-} = c_- + (1 - c_-)H$. This function belongs to $\mathcal{I}$. So does $F\psi_{c_-}$ since F maps increasing functions into increasing functions. Consequently, the limit $F\psi_{c_-}(-\infty)$ exists and by s-homogeneity, we have $F\psi_{c_-}(-\infty) = f(c_-) < c_-$. Therefore, there exists $v \in \mathbb{R}$ such that $F\psi_{c_-} \leqslant T^v\psi_{c_-}$. In addition, $J_{c_+}(\psi_{c_-}) = 0$ for every $c_+ \in (c, 1)$ and the statement follows.

(2) The property $T^{-v}F\psi \leqslant \psi$ implies the inequality $T^{-v}F\psi(-\infty) \leqslant \psi(-\infty)$. By s-homogeneity, we have $T^{-v}F\psi(-\infty) = f(\psi(-\infty))$. Consequently, $f(\psi(-\infty)) \leqslant \psi(-\infty)$ and $\psi(-\infty) < c_+ < 1$. According to the definition of f, we conclude that $\psi(-\infty) \leqslant c$.

Similarly, one can show that $f(\psi(+\infty)) \leqslant \psi(+\infty)$. The condition $\psi(+\infty) \geqslant c_+ > c$ imposes $\psi(+\infty) = 1$. □

The second statement of this lemma shows that the quantity

$$\bar{v} = \sup\{v \in \mathbb{R} : \mathcal{S}_{v,c_+} \neq \emptyset\}$$

does not depend on c_+, provided that the latter belongs to $(c, 1)$. The first statement shows that $\bar{v} > -\infty$. In order to obtain more results, we are going to consider the minimal function in $\mathcal{S}_{v,c_+}$.

2.2. Properties of sub-fronts

Given $v \in \mathbb{R}$ and $c_+ \in (c, 1)$, assume that $\mathcal{S}_{v,c_+}$ is non-empty. We define the function η_{v,c_+} by

$$\eta_{v,c_+}(x) = \inf_{\psi\in\mathcal{S}_{v,c_+}} \psi(x), \qquad x \in \mathbb{R}.$$

This function will be the starting point of the front shape construction. Its main properties are given in the following statement.

Lemma 2.2. *Let $c_+ \in (c, 1)$.*

(1) If $\mathcal{S}_{v,c_+} \neq \emptyset$, then $\eta_{v,c_+} \in \mathcal{S}_{v,c_+}$.
(2) If $\mathcal{S}_{v_2,c_+} \neq \emptyset$ and if $v_1 < v_2$, then $\mathcal{S}_{v_1,c_+} \neq \emptyset$ and $\eta_{v_1,c_+} \leqslant \eta_{v_2,c_+}$.
(3) $\bar{v} < +\infty$ and $\mathcal{S}_{\bar{v},c_+} \neq \emptyset$.

Proof. (1) Let $x < y \in \mathbb{R}$ and $\varepsilon > 0$. There exists $\psi \in \mathcal{S}_{v,c_+}$ such that

$$\eta_{v,c_+}(x) - \varepsilon \leqslant \psi(x) - \varepsilon \leqslant \psi(y) - \varepsilon \leqslant \eta_{v,c_+}(y).$$

Since ε is arbitrary, it follows that $\eta_{v,c_+}(x) \leqslant \eta_{v,c_+}(y)$ for all $x < y$ and consequently $\eta_{v,c_+} \in \mathcal{I}$.

For every $\psi \in \mathcal{S}_{v,c_+}$, we have $\eta_{v,c_+} \leqslant \psi$ and then by monotony $T^{-v}F\eta_{v,c_+} \leqslant T^{-v}F\psi \leqslant \psi$. Since ψ is arbitrary, we conclude that $T^{-v}F\eta_{v,c_+} \leqslant \eta_{v,c_+}$.

Finally, let $x > 0$ and $\varepsilon > 0$. There exists $\psi \in \mathcal{S}_{v,c_+}$ such that

$$\eta_{v,c_+}(-x) \leqslant \psi(-x) < c_+ \qquad \text{and} \qquad \psi(x) - \varepsilon \leqslant \eta_{v,c_+}(x).$$

Since ε is arbitrary, we conclude that $\eta_{v,c_+}(-x) < c_+ \leqslant \eta_{v,c_+}(x)$ for all $x > 0$, which implies that $J_{c_+}(\eta_{v,c_+}) = 0$. The first statement is proved.

(2) By monotony, we have $T^{-v_1}F\eta_{v_2,c_+} \leqslant T^{-v_2}F\eta_{v_2,c_+}$, which implies, since $\eta_{v_2,c_+} \in \mathcal{S}_{v_2,c_+}$, the inequality $T^{-v_1}F\eta_{v_2,c_+} \leqslant \eta_{v_2,c_+}$. Therefore, η_{v_2,c_+} belongs to $\mathcal{S}_{v_1,c_+}$ and by definition, we have $\eta_{v_1,c_+} \leqslant \eta_{v_2,c_+}$.

(3) We prove that $\bar{v} < +\infty$ by contradiction. Assume that $\mathcal{S}_{n,c_+}$ is not empty for all $n \in \mathbb{N}$. Then the previous statement and the monotony of F imply that $F\eta_{0,c_+} \leqslant T^n\eta_{n,c_+}$ for all $n \in \mathbb{N}$.

Lemma 2.1 tells us that $\eta_{0,c_+}(+\infty) = 1$. The same property holds for $F\eta_{0,c_+}$ by s-homogeneity. Hence there exists $x_{c_+} \in \mathbb{R}$ such that $F\eta_{0,c_+}(x_{c_+}) > c_+$. But since $J_{c_+}(\eta_{n,c_+}) = 0$, we have $\eta_{n,c_+}(x_{c_+} - n) < c_+$ for $n > x_{c_+}$ and hence a contradiction.

It remains to be proved that $\mathcal{S}_{\bar{v},c_+}$ is not empty. The s-homogeneity implies that

$$F\eta = \lim_{\substack{v\to\bar{v}\\ v<\bar{v}}} F\eta_{v,c_+} \qquad \text{where } \eta = \lim_{\substack{v\to\bar{v}\\ v<\bar{v}}} \eta_{v,c_+}.$$

The existence of the second limit is ensured by the monotony of η_{v,c_+} with v.

Applying lemma B.1, we obtain

$$\lim_{\substack{v\to\bar{v}\\ v<\bar{v}}} T^{-v}F\eta_{v,c_+}(x) = T^{-\bar{v}}F\eta(x)$$

for all x where $T^{-\bar{v}}F\eta$ is continuous. Consequently, the inequality $T^{-\bar{v}}F\eta(x) \leqslant \eta(x)$ holds for such points. These points form a dense subset of $\mathbb{R}$ because $F\eta$ is an increasing function. By s-homogeneity, we conclude that $T^{-\bar{v}}FP_\ell\eta \leqslant P_\ell\eta$. By taking the limit $x \to -\infty$, we obtain $f(\eta(-\infty)) \leqslant \eta(-\infty)$. Since $\eta(x) \leqslant c_+ < 1$ for all $x < 0$, it follows that $\eta(-\infty) \leqslant c$. On the other hand, $\eta(x) \geqslant c_+$ for all $x > 0$. We conclude that $J_{c_+}(\eta) \in \mathbb{R}$ and $T^{-J_{c_+}(\eta)}P_\ell\eta \in \mathcal{S}_{\bar{v},c_+}$. □

2.3. *Sub-fronts and super-fronts*

In the construction of fronts, we shall need other sub-fronts, namely increasing functions from $\mathbb{R}$ to $[c, 1]$. These solutions belong to the set $\mathcal{D}_{v,c_+} = \{\psi \in \mathcal{S}_{v,c_+} : \psi(-\infty) = c\}$. The arguments of the proof of the second statement of lemma 2.1 show that any $\psi \in \mathcal{D}_{v,c_+}$ satisfies $\psi(-\infty) = c$ and $\psi(+\infty) = 1$. Consequently, the quantity

$$\bar{v}_s = \sup\{v \in \mathbb{R} : \mathcal{D}_{v,c_+} \neq \emptyset\}$$

does not depend on c_+, provided that the latter belongs to $(c, 1)$, and since $\mathcal{D}_{v,c_+} \subset \mathcal{S}_{v,c_+}$, we have $\bar{v}_s \leqslant \bar{v}$. As we shall see in proposition 3.3, it may happen that $\bar{v}_s = -\infty$.

In addition to sub-fronts, we introduce increasing functions from $\mathbb{R}$ to $[0, c]$ which are super-fronts. Given $c_- \in (0, c)$ and $v \in \mathbb{R}$, consider the set

$$\mathcal{E}_{v,c_-} = \{\psi \in \mathcal{I} : T^v\psi \leqslant F\psi,\ J_{c_-}(\psi) = 0 \text{ and } \psi(+\infty) = c\}.$$

An analysis similar to the one done for sub-fronts shows that the quantity

$$\bar{v}_i = \inf\{v \in \mathbb{R} : \mathcal{E}_{v,c_-} \neq \emptyset\}$$

does not depend on c_-, provided that the latter belongs to $(0, c)$ and we have $\bar{v}_i > -\infty$.

These velocities satisfy the following inequality.

Lemma 2.3. *Given a non-degenerate distribution function h and a bistable map f, we have $\bar{v}_s < \bar{v}_i$.*

In the proof of theorem 1.2, we shall see that $\bar{v}_s = \bar{v}_i$ when h is degenerate.

Proof. We suppose that both $\bar{v}_s$ and $\bar{v}_i$ are finite. Otherwise the result follows trivially from the inequalities $\bar{v}_s < +\infty$ and $\bar{v}_i > -\infty$. By using arguments similar to those in the proof of lemma 2.2, one proves the existence of a function $\psi \in \mathcal{D}_{\bar{v}_s,c_+}$. Since ψ is above c, we have $\psi \leqslant f \circ \psi$ and then $h_{\bar{v}_s} * \psi \leqslant \psi$, where $h_{\bar{v}_s} = T^{-\bar{v}_s}h$. Statement 1 of lemma A.3 then implies that $\lim_{n\to\infty} h_{\bar{v}_s}^{*n} = 0$.

Similarly, one proves the existence of $\varphi \in \mathcal{E}_{\bar{v}_i,c_-}$ and we have $\varphi \leqslant h_{\bar{v}_i} * \varphi$, where $h_{\bar{v}_i} = T^{-\bar{v}_i}h$. Statement 2 of lemma A.3 then implies that $\lim_{n\to\infty} h_{\bar{v}_i}^{*n} = 1$.

The distinct limits of n-fold convolutions imply that $\bar{v}_s \neq \bar{v}_i$. If we assume that $\bar{v}_i < \bar{v}_s$, the monotony of h gives $h_{\bar{v}_i}^{*n} \leqslant h_{\bar{v}_s}^{*n}$, which is in contradiction with the previous limits of n-fold convolutions. □

3. Construction of fronts

3.1. General construction

Using minimal sub-fronts, we can now construct increasing solutions of the front equation. Functions resulting from this construction may be above c. But, as we shall see, this situation can be prevented and we have the following condition for the existence of fronts.

Theorem 3.1. *Let h be a distribution function and let f be a bistable map. If $\bar{v}_s < \bar{v}$ then the map F has fronts of velocity $\bar{v}$.*

Before starting the proof, we present an auxiliary result.

Lemma 3.2. *Let $\{\alpha_n\}_{n\in\mathbb{N}}$ be a sequence of real numbers satisfying the following property for all $m \in \mathbb{N}$:*

$$\liminf_{n\to\infty} (\alpha_{n+m} - \alpha_n) = mv,$$

where v does not depend on m. Then, there exists a strictly increasing sequence $\{n_k\}_{k\in\mathbb{N}}$ (which is independent of m) such that, for all $m \in \mathbb{N}$, we have

$$\lim_{k\to\infty} (\alpha_{n_k+m} - \alpha_{n_k}) = mv.$$

Proof. By replacing α_n by $\alpha_n - nv$, we can always assume that $v = 0$. Given $n \in \mathbb{N}$, let

$$\beta_n = \inf_{k\geqslant n} (\alpha_{k+1} - \alpha_k) \qquad \text{and} \qquad \gamma_n = \alpha_n - \sum_{k=0}^{n-1} \beta_k.$$

By assumption on the sequence $\{\alpha_n\}$, we have $\lim_{n\to\infty}\sum_{k=n}^{n+m-1}\beta_k = 0$ for all $m \in \mathbb{N}$, and then

$$\liminf_{n\to\infty}(\gamma_{n+m} - \gamma_n) = \liminf_{n\to\infty}(\alpha_{n+m} - \alpha_n) = 0, \qquad m \in \mathbb{N}.$$

Moreover, the sequence $\{\gamma_n\}$ is increasing because $\gamma_{n+1} - \gamma_n = \alpha_{n+1} - \alpha_n - \beta_n \geqslant 0$. As a consequence, one proves by induction that for each $m \in \mathbb{N}$ there exists an increasing sequence $\{n_k^m\}_{k\in\mathbb{N}}$ such that

$$0 \leqslant \gamma_{n_k^m+m} - \gamma_{n_k^m} \leqslant \frac{1}{k} \qquad \text{and} \qquad n_k^m > n_{k-1}^{m-1}, \qquad k, m \in \mathbb{N}.$$

The diagonal sequence $\{n_k^k\}_{k\in\mathbb{N}}$ is the desired sequence. Indeed, we have

$$0 \leqslant \lim_{k\to\infty}(\alpha_{n_k^k+m} - \alpha_{n_k^k}) = \lim_{k\to\infty}(\gamma_{n_k^k+m} - \gamma_{n_k^k}) \leqslant \lim_{k\to\infty}(\gamma_{n_k^k+k} - \gamma_{n_k^k}) = 0, \qquad m \in \mathbb{N}. \qquad \square$$

Proof of theorem 3.1. Let $c_+ \in (c, 1)$. By lemma 2.2, the function $\eta_{\bar{v},c_+}$, which we denote by η, exists. Given $n \in \mathbb{N}$, let j_n denote the quantity $J_{c_+}(F^n\eta)$. This quantity belongs to $\mathbb{R}$ for all n since by s-homogeneity and the properties of f, we have $F^n\eta(-\infty) = f^n(\eta(-\infty)) \leqslant c$ and $F^n\eta(+\infty) = 1$.

We consider the functions ϕ_n defined by

$$\phi_n(x) = \inf_{k\geqslant n}\{T^{-j_k}F^k\eta(x)\}, \qquad x \in \mathbb{R}.$$

We have $\phi_n \leqslant \phi_{n+1}$; thus, the following limit exists: $\phi_\infty = \lim_{n\to\infty}\phi_n$.

Given $m \in \mathbb{N}$, let $\beta_m = \liminf_{n\to\infty}(j_{n+m} - j_n)$. We are going to prove that $\beta_m = m\bar{v}$ for all m.

We have $F^{n+1}\eta \leqslant T^{\bar{v}}F^n\eta$, which implies the inequality $j_{n+1} \geqslant j_n + \bar{v}$ and then $\beta_m \geqslant m\bar{v}$ for all m.

In order to prove the converse inequality, we first observe that $T^{-j_n}F^n\eta \in \mathcal{S}_{\bar{v},c_+}$ for all n. By introducing the function $\psi_{c_+} = c_+P_\ell H$, we have $\psi_{c_+} \leqslant T^{-j_n}F^n\eta$. Consequently, $F\psi_{c_+} \leqslant T^{-j_n}F^{n+1}\eta$ and then

$$J_{c_+}(F\psi_{c_+}) \geqslant J_{c_+}(T^{-j_n}F^{n+1}\eta) = j_{n+1} - j_n.$$

In other words, $j_{n+1} - j_n$ is uniformly bounded from above and hence $\beta_m < +\infty$ for all m.

Given $m \geqslant 1$ and $\varepsilon > 0$, let $n_{m,\varepsilon}$ be such that for all $k \geqslant n_{m,\varepsilon}$, we have $\beta_m - \varepsilon \leqslant j_{k+m} - j_k$. By definition of ϕ_n, given $n \geqslant n_{m,\varepsilon}$, we have $\phi_n \leqslant T^{-j_k}F^k\eta$ for all $k \geqslant n$ and then

$$T^{-(\beta_m-\varepsilon)}F^m\phi_n \leqslant T^{-(j_{k+m}-j_k)}F^m\phi_n \leqslant T^{-j_{k+m}}F^{k+m}\eta, \qquad k \geqslant n.$$

Consequently, $T^{-(\beta_m-\varepsilon)}F^m\phi_n \leqslant \phi_{n+m}$. By s-homogeneity, one can take the limit $n \to \infty$ and then the limit $\varepsilon \to 0$ and apply P_ℓ to obtain

$$P_\ell T^{-\beta_m}F^m\phi_\infty \leqslant P_\ell\phi_\infty$$

and by s-homogeneity, $T^{-\beta_m}F^mP_\ell\phi_\infty \leqslant P_\ell\phi_\infty$.

Consider the function defined by $\varphi_m(x) = \min_{0\leqslant k<m}\{T^{-k\beta_m/m}F^kP_\ell\phi_\infty(x)\}$ for all $x \in \mathbb{R}$. We have $\phi_\infty(-\infty) \leqslant c$ and $\phi_\infty(+\infty) = 1$. By s-homogeneity, it follows that $\varphi_m(-\infty) \leqslant c$ and $\varphi_m(+\infty) = 1$, which implies that $J_{c_+}(\varphi_m) \in \mathbb{R}$.

Now, by monotony of F, we have

$$T^{-\beta_m/m}F\varphi_m \leqslant T^{-(k+1)\beta_m/m}F^{k+1}P_\ell\phi_\infty, \qquad 0 \leqslant k < m.$$

Choosing $k = m-1$, we obtain $T^{-\beta_m/m}F\varphi_m \leqslant T^{-\beta_m}F^mP_\ell\phi_\infty \leqslant P_\ell\phi_\infty$ and consequently $T^{-\beta_m/m}F\varphi_m \leqslant \varphi_m$. It follows that $\mathcal{S}_{\beta_m/m,c_+}$ is not empty and consequently $\beta_m \leqslant m\bar{v}$, which leads to the desired property.

Since $\beta_m = m\bar{v}$, one can use lemma 3.2 to state the existence of a strictly increasing sequence $\{n_k\}$ such that for all $m \in \mathbb{N}$

$$\beta_m = \lim_{k\to\infty} (j_{n_k+m} - j_{n_k}).$$

The sequence $\{T^{-j_{n_k}} F^{n_k}\eta\}_{k\in\mathbb{N}}$ is composed of increasing functions with values in [0, 1]. By Helly's selection theorem (see chapter 10 in [12] or exercise 13, chapter 7 in [15]), it has a pointwise convergent subsequence. Without loss of generality, let

$$\eta_\infty = \lim_{k\to\infty} T^{-j_{n_k}} F^{n_k}\eta.$$

Then η_∞ is an increasing function satisfying $\eta_\infty(x) \geqslant c_+$ and $\eta_\infty(-x) \leqslant c_+$ for all $x > 0$. For all k, we have

$$\phi_{n_k} \leqslant T^{-j_{n_k+m}} F^{n_k+m}\eta = T^{-(j_{n_k+m}-j_{n_k})} F^m T^{-j_{n_k}} F^{n_k}\eta, \qquad m \in \mathbb{N}.$$

By s-homogeneity and by lemma B.1, this implies (firstly at the points where $T^{-m\bar{v}} F^m \eta_\infty$ is continuous and then by applying P_ℓ)

$$P_\ell \phi_\infty \leqslant T^{-m\bar{v}} F^m P_\ell \eta_\infty$$

and consequently $P_\ell\phi_\infty \leqslant T^{-m\bar{v}} F^m \eta_\infty$. Furthermore, $\eta \in \mathcal{S}_{\bar{v},c_+}$ implies the inequality $T^{-\bar{v}} F T^{-j_{n_k}} F^{n_k}\eta \leqslant T^{-j_{n_k}} F^{n_k}\eta$. The limit $k \to \infty$ gives $T^{-\bar{v}} F\eta_\infty \leqslant \eta_\infty$ and then

$$T^{-(m+1)\bar{v}} F^{m+1}\eta_\infty \leqslant T^{-m\bar{v}} F^m \eta_\infty, \qquad m \in \mathbb{N}.$$

These inequalities imply the existence of the following function:

$$\phi = P_r(\lim_{m\to\infty} T^{-m\bar{v}} F^m \eta_\infty).$$

This function is increasing and right continuous. Using s-homogeneity, one proves that $T^{-\bar{v}} F\phi = \phi$.

To prove that ϕ is a distribution function, it remains to show the appropriate asymptotic behaviour. By using s-homogeneity in the previous relation, one shows that $\phi(-\infty)$ and $\phi(+\infty)$ are fixed points of f. Moreover, the inequalities of the previous paragraph imply that $P_\ell\phi_\infty \leqslant \phi \leqslant \eta_\infty$ and then $\phi(x) \geqslant c_+$ and $\phi(-x) \leqslant c_+$ for all $x > 0$. Consequently, $\phi(+\infty) = 1$ and $\phi(-\infty) \in \{0, c\}$. If $\phi(-\infty) = c$, then $\mathcal{D}_{\bar{v},c_+}$ would not be empty and we would have a contradiction with the assumption $\bar{v}_s < \bar{v}$. So $\phi(-\infty) = 0$ and there exist fronts of velocity $\bar{v}$. □

3.2. Existence of fronts in special cases

We now provide conditions on the local map and on the coupling to ensure that $\bar{v}_s = -\infty$. Since $\bar{v} > -\infty$, we are sure that fronts exist in such systems.

To that purpose, we consider the upper right derivative of f at c, say $\bar{f}'_r(c)$, and the infimum bound of the support of h,

$$v_{\min}(h) = \inf\{x \in \mathbb{R} : h(x) > 0\}.$$

Proposition 3.3. *If f is a bistable map such that $\bar{f}'_r(c) = +\infty$ and if h is a distribution function such that $v_{\min}(h) = -\infty$, then $\bar{v}_s = -\infty$.*

This result is a consequence of the following statement.

Lemma 3.4. *Given $c_+ \in (c, 1)$, consider the function $\psi_{c_+} = c + (c_+ - c)P_\ell H$. We have*

(1) $\bar{v}_s \leqslant J_{c_+}(F\psi_{c_+})$,
(2) $P_\ell h(J_{c_+}(F\psi_{c_+})) \leqslant (c_+ - c)/(f(c_+) - c)$.

Proof. (1) Assume that $\mathcal{D}_{v,c_+}$ is not empty and let $\psi \in \mathcal{D}_{v,c_+}$. Then $\psi_{c_+} \leqslant \psi$ and hence $F\psi_{c_+} \leqslant F\psi \leqslant T^v\psi$. This implies that $J_{c_+}(F\psi_{c_+}) \geqslant v$ and, since $v \leqslant \bar{v}_s$ is arbitrary, we obtain the desired conclusion.

(2) An explicit calculation shows that $F\psi_{c_+}(x) = c + (f(c_+) - c)P_\ell h(x)$ for all $x \in \mathbb{R}$. By definition of $J_{c_+}(F\psi_{c_+})$, we then have for every $\delta > 0$

$$c + (f(c_+) - c)P_\ell h(J_{c_+}(F\psi_{c_+}) - \delta) < c_+.$$

However, $f(c_+)$ is larger than c and δ is arbitrary. Therefore, we obtain the desired conclusion. □

Proof of proposition 3.3. Since $\bar{v}_s$ does not depend on c_+, lemma 3.4 shows that

$$P_\ell h(\bar{v}_s) \leqslant \liminf_{\substack{c_+ \to c \\ c_+ > c}} \frac{c_+ - c}{f(c_+) - c} = \frac{1}{\bar{f}_r'(c)} = 0,$$

from which it results that $\bar{v}_s \leqslant v_{\min}(h) = -\infty$ and the proposition follows. □

3.3. Existence of fronts using approximations

In the proof of existence in theorem 1.1 as well as in the proof of the continuity of front velocity, we shall employ the following result on the existence of fronts for the limit of a convergent sequence of systems of the form $h * f$.

Theorem 3.5. *Let $\{f_n\}_{n\in\mathbb{N}}$ be a sequence of bistable maps such that* $\lim_{n\to\infty} f_n = f$, *where f is a bistable map and let $\{h_n\}_{n\in\mathbb{N}}$ be a sequence of distribution functions such that* $\lim_{n\to\infty} d(h_n, h) = 0$, *where h is a non-degenerate distribution function. Assume that, for each n, the map $F_n = h_n * f_n$ has fronts of velocity v_n and suppose that* $\lim_{n\to\infty} v_n = v$. *Then, the map $F = h * f$ has fronts of velocity v.*

The proof uses the following statement.

Lemma 3.6. *Let $\{\psi_n\}_{n\in\mathbb{N}}$ be a sequence of distribution functions such that* $\lim_{n\to\infty} \psi_n = \psi$ *and let $\{f_n\}_{n\in\mathbb{N}}$ be a sequence of increasing maps defined on* $[0, 1]$ *such that* $\lim_{n\to\infty} f_n = f$, *where f is continuous. Then* $\lim_{n\to\infty} f_n \circ \psi_n = f \circ \psi$.

Proof. The uniform continuity of f and the monotony of the f_n imply that the convergence of the sequence $\{f_n\}$ holds in the uniform topology. The proof of this claim can be done using arguments similar to those used in the proof of lemma B.3 and is left to the reader.

Fix $x \in \mathbb{R}$ and $\varepsilon > 0$. Let n_0 be such that $n > n_0$ implies $\|f_n - f\| < \varepsilon/2$ and let n_1 be such that $n > n_1$ implies $|f \circ \psi_n(x) - f \circ \psi(x)| < \varepsilon/2$. For all n larger than n_0 and n_1, we have

$$|f_n \circ \psi_n(x) - f \circ \psi(x)| \leqslant |f_n \circ \psi_n(x) - f \circ \psi_n(x)| + |f \circ \psi_n(x) - f \circ \psi(x)| < \varepsilon,$$

which is the desired conclusion. □

Proof of theorem 3.5. Fix $c_- \in (0, c)$ and $c_+ \in (c, 1)$ and recall the definition of $\bar{v}_s$ from section 2.3. Either $v \leqslant \bar{v}_s$ or $v > \bar{v}_s$. We prove that in both cases, the map $F = h * f$ has fronts of velocity v.

Assume that $v \leqslant \bar{v}_s$, let ϕ_n be a shape of fronts of velocity v_n for the map $F_n = h_n * f_n$ and let $j_n = J_{c_-}(\phi_n) \in \mathbb{R}$. By Helly's selection theorem, the sequence $\{T^{-j_n}\phi_n\}$ has a pointwise convergent subsequence that we assume to be that sequence, for the sake of notation. Let

$$\phi_\infty = \lim_{n\to\infty} T^{-j_n}\phi_n \qquad \text{and} \qquad \phi = P_r\phi_\infty.$$

We prove that ϕ is a distribution function that satisfies $F\phi = T^v\phi$. This function ϕ belongs to $\mathcal{I}$ and is right continuous. Moreover, using lemma B.1 and applying P_r, we obtain

$$P_r(\lim_{n\to\infty} T^{v_n}T^{-j_n}\phi_n) = T^v\phi.$$

From lemma 3.6, we have

$$\lim_{n\to\infty} f_n \circ T^{-j_n}\phi_n = f \circ \phi_\infty$$

and then, by proposition B.2 and lemma B.3, we conclude that

$$\lim_{n\to\infty} h_n * f_n \circ T^{-j_n}\phi_n(x) = h * f \circ \phi_\infty(x)$$

for all x, where $h * f \circ \phi_\infty$ is continuous. Applying P_r, it follows by s-homogeneity that $h * f \circ \phi = T^v\phi$.

It remains to be shown that ϕ has the appropriate asymptotic behaviour. The relation $F\phi = T^v\phi$ implies that $\phi(-\infty)$ and $\phi(+\infty)$ are fixed points of f. In addition, we have $\phi(x) \leqslant c_-$ for every $x < 0$ and then $\phi(-\infty) = 0$. On the other hand $\phi(0) \geqslant c_-$ and then $\phi(+\infty) \in \{c, 1\}$. The assumption $v \leqslant \bar{v}_s$ and lemma 2.3 imply that $v < \bar{v}_i$, which ensures $\phi(+\infty) = 1$.

Assume now that $\bar{v}_s < v$. The argument is similar. Given $n \in \mathbb{N}$, let $j_n = J_{c_+}(\phi_n)$ and let again

$$\phi_\infty = \lim_{n\to\infty} T^{-j_n}\phi_n \qquad \text{and} \qquad \phi = P_r\phi_\infty.$$

This function ϕ is increasing and right continuous and satisfies the relation $F\phi = T^v\phi$. Moreover, $\phi(0) \geqslant c_+$ and then $\phi(+\infty) = 1$. In addition, we have $\phi(-\infty) \in \{0, c\}$ and the inequality $\bar{v}_s < v$ imposes $\phi(-\infty) = 0$. □

4. Proof of the main results

This section contains the proofs of theorems 1.1, 1.2 and 1.3. Some proofs are accomplished by using approximation techniques and use the following relations between f, h and the velocity, $\bar{v}$.

Lemma 4.1. *Given a bistable map f and a distribution function h, let $c_- \in (0, c)$ and $c_+ \in (c, 1)$. The following inequalities hold:*

$$\frac{c_- - f(c_-)}{1 - f(c_-)} \leqslant h(\bar{v}) \qquad \textit{and} \qquad P_\ell h(\bar{v}) \leqslant \frac{c_+}{f(c_+)}.$$

Proof. The second inequality is proved using arguments similar to those used in the proof of lemma 3.4. Consider the function $\psi_{c_+} = c_+P_\ell H$. One shows that $\bar{v} \leqslant J_{c_+}(F\psi_{c_+})$ and $P_\ell h(J_{c_+}(F\psi_{c_+})) \leqslant c_+/f(c_+)$, from which the inequality follows.

In order to prove the first inequality, consider the function $\psi_{c_-} = c_- + (1 - c_-)H$. We have $F\psi_{c_-} = f(c_-) + (1 - f(c_-))h$ and then $J_{c_-}(F\psi_{c_-}) \in \mathbb{R}$. In addition, the definition of ψ_{c_-} shows that $F\psi_{c_-} \leqslant T^{J_{c_-}(F\psi_{c_-})}\psi_{c_-}$. Since $J_{c_-}(\psi_{c_-}) = 0$, it follows that $\psi_{c_-} \in \mathcal{S}_{J_{c_-}(F\psi_{c_-}),c_+}$ and then $J_{c_-}(F\psi_{c_-}) \leqslant \bar{v}$.

The right continuity of h and the definition of $J_{c_-}(F\psi_{c_-})$ show that $(c_- - f(c_-))/(1 - f(c_-)) \leqslant h(J_{c_-}(F\psi_{c_-}))$. The first inequality follows from the monotony of h. □

4.1. Proof of existence of fronts

Theorem 1.1 claims the existence of fronts and, when f is regular, the uniqueness of their velocity. For the sake of clarity, we only prove existence in this subsection. The proof of uniqueness is postponed to the next subsection.

If $h = T^v H$ for some $v \in \mathbb{R}$, then $FH = T^v H$ and the existence of fronts is proved. Therefore, in the rest of this subsection, h is assumed not to be degenerate.

We want to apply theorem 3.5. For this purpose, we construct a family $\{F_\varepsilon\}$ of approximations of F that satisfy the condition of proposition 3.3.

Let $c_- \in (0, c)$ and $c_+ \in (c, 1)$ be fixed. We introduce the following numbers:

$$v^- = \inf\left\{v \in \mathbb{R} : \frac{c_- - f(c_-)}{1 - f(c_-)} \leqslant h(v)\right\} \qquad \text{and} \qquad v^+ = \sup\left\{v \in \mathbb{R} : P_\ell h(v) \leqslant \frac{c_+}{f(c_+)}\right\}.$$

Now, given $\varepsilon \in (0, \sqrt{c_+ - c})$, consider the map f_ε defined by

$$f_\varepsilon(x) = \begin{cases} f(x) & \text{if } x \in [0, c], \\ \max\{c + \varepsilon\sqrt{x - c}, f(x)\} & \text{if } x \in (c, 1] \end{cases}$$

and given $\varepsilon \in (0, (c_- - f(c_-))/(1 - f(c_-)))$, consider the distribution function defined by

$$h_\varepsilon(x) = \begin{cases} \max\{\varepsilon e^{x - v^-}, h(x)\} & \text{if } x < v^-, \\ h(x) & \text{if } x \geqslant v^-. \end{cases}$$

Let $\varepsilon_0 = \min\{\sqrt{c_+ - c}, (c_- - f(c_-))/(1 - f(c_-))\}$ and let $\{F_\varepsilon\}_{0<\varepsilon<\varepsilon_0}$ be the family of mappings in $\mathcal{B}$ defined by $F_\varepsilon = h_\varepsilon * f_\varepsilon$.

We have $\bar{f}'_{\varepsilon_r}(c) = +\infty$ and $v_{\min}(h_\varepsilon) = -\infty$. According to proposition 3.3 and theorem 3.1, for every $\varepsilon \in (0, \varepsilon_0)$, there exist a distribution function ϕ_ε and a real number v_ε such that $F_\varepsilon \phi_\varepsilon = T^{v_\varepsilon} \phi_\varepsilon$.

Moreover, $\lim_{\varepsilon \to 0} f_\varepsilon = f$ and $\lim_{\varepsilon \to 0} h_\varepsilon = h$. In order to apply theorem 3.5, we show that the velocities v_ε are bounded.

Lemma 4.2. *For every* $0 < \varepsilon < \varepsilon_0$, *we have* $v^- \leqslant v_\varepsilon \leqslant v^+$.

Proof. From the definition of f_ε, we have $f_\varepsilon(c_-) = f(c_-)$ and $f_\varepsilon(c_+) = f(c_+)$ for every $\varepsilon \in (0, \varepsilon_0)$. By using lemma 4.1, we obtain the following inequalities for all $\varepsilon \in (0, \varepsilon_0)$:

$$\frac{c_- - f(c_-)}{1 - f(c_-)} \leqslant h_\varepsilon(v_\varepsilon) \qquad \text{and} \qquad P_\ell h_\varepsilon(v_\varepsilon) \leqslant \frac{c_+}{f(c_+)}.$$

Moreover, from the definition of h_ε and since $\varepsilon < (c_- - f(c_-))/(1 - f(c_-))$, we have $h_\varepsilon(x) < (c_- - f(c_-))/(1 - f(c_-))$ for all $x < v^-$. Together with the previous first inequality, this implies that $v_\varepsilon \geqslant v^-$.

Furthermore, the condition $h_\varepsilon(x) = h(x)$ for all $x \geqslant v^-$ implies that $P_\ell h(v_\varepsilon) \leqslant c_+/f(c_+)$ and, from the definition of v^+, we conclude that $v_\varepsilon \leqslant v^+$. □

By lemma 4.2, there exists a sequence $\{\varepsilon_n\}_{n\in\mathbb{N}}$ of elements in $(0, \varepsilon_0)$ such that the following limits exist:

$$\lim_{n\to\infty} \varepsilon_n = 0 \qquad \text{and} \qquad \lim_{n\to\infty} v_{\varepsilon_n} = v,$$

for some $v \in [v^-, v^+]$. One can now apply theorem 3.5 with the sequences $\{f_{\varepsilon_n}\}$ and $\{h_{\varepsilon_n}\}$ to obtain the existence of fronts of velocity v for the map $F = h * f$.

4.2. Proof of uniqueness of the velocity

In this subsection, we prove the second part of theorem 1.1, namely the uniqueness of the front velocity when f is regular. The following result will be used.

Lemma 4.3. *Assume that the bistable map f is regular and let ϕ be a shape of fronts of velocity v. For every $c_+ \in (1-\delta, 1)$, where δ is the number of the condition of regularity of f, there exists $s \in \mathbb{R}$ such that $\phi \leqslant T^{-s}\eta_{v,c_+}$.*

Proof. Assume that $J_{c_+}(\phi) = 0$. Note that another choice of $J_{c_+}(\phi)$ would only affect the value of s in the statement. Let $c_- \in (0, \delta)$ and let $j = J_{c_-}(\phi) \leqslant 0$. Since $\eta_{v,c_+}(x) \geqslant c_+$ for every $x > 0$, there exists $s \geqslant 0$ such that

$$\phi(x) \leqslant T^{-s}\eta_{v,c_+}(x) \qquad \text{if } x \in [j, 0].$$

Consider the family of functions $\{\psi_n\}_{n\in\mathbb{N}}$ defined as follows:

$$\psi_0(x) = \begin{cases} \eta_{v,c_+}(x) & \text{if } x \notin [j, 0], \\ \phi(x) & \text{if } x \in [j, 0], \end{cases}$$

$$\psi_{n+1}(x) = \begin{cases} \max\{\psi_n(x), T^{-v}F\psi_n(x)\} & \text{if } x \notin [j, 0], \\ \phi(x) & \text{if } x \in [j, 0]. \end{cases}$$

We prove using induction that $\psi_n \leqslant \phi$ and $\psi_n \leqslant T^{-s}\eta_{v,c_+}$ for all $n \in \mathbb{N}$.

By construction and since $\eta_{v,c_+} \leqslant \phi$ and $s \geqslant 0$, these inequalities hold for $n = 0$. Assume they hold for some $n \in \mathbb{N}$. Then we have

$$T^{-v}F\psi_n \leqslant T^{-v}F\phi = \phi \qquad \text{and} \qquad T^{-v}F\psi_n \leqslant T^{-v}FT^{-s}\eta_{v,c_+} \leqslant T^{-s}\eta_{v,c_+},$$

which imply that the inequalities hold for $n+1$.

By using once again the definition of ψ_n, we obtain the inequalities $\eta_{v,c_+} \leqslant \psi_n \leqslant \psi_{n+1} \leqslant \phi$ for all n. Thus the following limit exists,

$$\psi_\infty = \lim_{n\to\infty} \psi_n$$

and satisfies the inequalities $\eta_{v,c_+} \leqslant \psi_\infty$, $\psi_\infty \leqslant \phi$ and $\psi_\infty \leqslant T^{-s}\eta_{v,c_+}$. More precisely, we have

$$\begin{cases} \psi_\infty(x) \leqslant \phi(x) < c_- & \text{if } x < j, \\ \psi_\infty(x) = \phi(x) & \text{if } x \in [j, 0], \\ c_+ \leqslant \eta_{v,c_+}(x) \leqslant \psi_\infty(x) \leqslant \phi(x) & \text{if } 0 < x. \end{cases} \tag{1}$$

Therefore, the monotony and regularity of f imply the inequality

$$f \circ \phi - f \circ \psi_\infty \leqslant \phi - \psi_\infty.$$

On the other hand, since $T^{-v}F\psi_n \leqslant \phi$, the definition of ψ_n implies that $T^{-v}F\psi_n \leqslant \psi_{n+1}$ and then by s-homogeneity $T^{-v}F\psi_\infty \leqslant \psi_\infty$.

As a consequence, we obtain

$$\begin{aligned} \phi - \psi_\infty &\leqslant T^{-v}F\phi - T^{-v}F\psi_\infty \\ &\leqslant T^{-v}h * (f \circ \phi - f \circ \psi_\infty) \\ &\leqslant T^{-v}h * (\phi - \psi_\infty). \end{aligned}$$

If $T^{-v}h \neq H$, then lemma A.4 implies that $\psi_\infty = \phi$. Since $\psi_\infty \leqslant T^{-s}\eta_{v,c_+}$, we obtain the desired result. If $T^{-v}h = H$, then the front shape equation imposes $f \circ \phi = \phi$ and the previous inequalities result in $f \circ \phi - f \circ \psi_\infty = \phi - \psi_\infty$. It follows that $f \circ \psi_\infty = \psi_\infty$, and by (1) we also conclude that $\phi = \psi_\infty \leqslant T^{-s}\eta_{v,c_+}$. □

We can now state and prove the uniqueness of the front velocity.

Theorem 4.4. *Assume that the bistable map f is regular. Then, the velocity of fronts is unique and is equal to $\bar{v}$, the quantity introduced in section 2.1.*

Proof. By definition, for any front velocity v, there exists a shape in $\mathcal{S}_{v,c_+}$ for any $c_+ \in (c, 1)$. Consequently, any front velocity satisfies the inequality $v \leqslant \bar{v}$.

By contradiction, assume the existence of a front velocity $v < \bar{v}$ and let ϕ be the corresponding shape and fix $c_+ \in (1-\delta, 1)$. Using statement 2 of lemmas 2.2 and 4.3, one obtains the existence of $s \in \mathbb{R}$ such that

$$\phi \leqslant T^{-s}\eta_{v,c_+} \leqslant T^{-s}\eta_{\bar{v},c_+}.$$

Applying $T^{-n\bar{v}}F^n$ to these inequalities, we obtain by monotony

$$T^{n(v-\bar{v})}\phi \leqslant T^{-s}\eta_{\bar{v},c_+}, \qquad n \in \mathbb{N}$$

and by taking the limit $n \to \infty$, we get $1 \leqslant T^{-s}\eta_{\bar{v},c_+}$, which is in contradiction with the fact that $\eta_{\bar{v},c_+} \in \mathcal{S}_{\bar{v},c_+}$, statements 1 and 3 of lemma 2.2. □

4.3. Proof of continuity of the front velocity

Theorem 1.2 claims that the sequence $\{v(f_n, h_n)\}$ of front velocities converges to $v(f, h)$, provided that the sequence $\{f_n\}$ converges to f and the sequence $\{h_n\}$ converges to h. Its proof follows the same lines as the proof of the existence of fronts. We denote by v_n the velocity $v(f_n, h_n)$. As in lemma 4.2, we prove that the sequence $\{v_n\}$ is bounded.

Let $c_- \in (0, c)$, $c_+ \in (c, 1)$ and $\varepsilon > 0$ be fixed such that

$$\varepsilon < \min\left\{\frac{c_- - f(c_-)}{1 - f(c_-)}, 1 - \frac{c_+}{f(c_+)}\right\}.$$

By assumption on the sequence $\{f_n\}$, let $n_0 \in \mathbb{N}$ be such that $n > n_0$ implies

$$\frac{c_- - f(c_-)}{1 - f(c_-)} - \frac{\varepsilon}{2} \leqslant \frac{c_- - f_n(c_-)}{1 - f_n(c_-)} \qquad \text{and} \qquad \frac{c_+}{f_n(c_+)} \leqslant \frac{c_+}{f(c_+)} + \frac{\varepsilon}{2}.$$

By applying lemma 4.1, we deduce that, for every $n > n_0$, we have

$$\frac{c_- - f(c_-)}{1 - f(c_-)} - \frac{\varepsilon}{2} \leqslant h_n(v_n) \qquad \text{and} \qquad P_\ell h_n(v_n) \leqslant \frac{c_+}{f(c_+)} + \frac{\varepsilon}{2}.$$

By assumption on the sequence $\{h_n\}$, there exists $n_1 \geqslant n_0$ such that for all $n > n_1$, we have

$$h_n(v_n) \leqslant h\left(v_n + \frac{\varepsilon}{2}\right) + \frac{\varepsilon}{2} \qquad \text{and} \qquad P_\ell h\left(v_n - \frac{\varepsilon}{2}\right) - \frac{\varepsilon}{2} \leqslant P_\ell h_n(v_n).$$

Consequently, for all $n > n_1$, we have

$$\frac{c_- - f(c_-)}{1 - f(c_-)} - \varepsilon \leqslant h\left(v_n + \frac{\varepsilon}{2}\right) \qquad \text{and} \qquad P_\ell h\left(v_n - \frac{\varepsilon}{2}\right) \leqslant \frac{c_+}{f(c_+)} + \varepsilon,$$

which proves that the sequence $\{v_n\}$ is bounded.

Let now $\{v_{n_i}\}_{i\in\mathbb{N}}$ be any convergent subsequence and let $v = \lim_{i\to\infty} v_{n_i}$ be its limit.

If h is not degenerate, then by applying theorem 3.5, we deduce that the map $F = h * f$ has fronts of velocity v. By uniqueness, we have $v = v(f, h)$ and since the subsequence is arbitrary, we conclude that $\lim_{n\to\infty} v(f_n, h_n) = v(f, h)$.

If $h = T^{v'}H$ for some $v' \in \mathbb{R}$, then $F\psi_c = T^{v'}\psi_c$, where $\psi_c = c + (1-c)H$. For any increasing function $\psi : \mathbb{R} \to [c, 1]$, we have $F\psi \geqslant T^{v'}\psi$. It follows that $\bar{v}_s = v'$. Similarly, one shows that $\bar{v}_i = v'$. If $v < \bar{v}_i$, then the same arguments as those in the proof of theorem 3.5 show the existence of fronts of velocity v. However, this is impossible by the uniqueness of front velocity. Similarly, the assumption $v > \bar{v}_s$ also leads to a contradiction. Therefore, we conclude that $\lim_{n\to\infty} v(f_n, h_n) = v(f, h)$ and theorem 1.2 is proved.

4.4. Proof of the existence of the velocity of interfacial orbits

In this subsection, we prove theorem 1.3 on the existence and uniqueness of the velocities of orbits whose initial condition is an interface.

We recall that a function $u \in \mathcal{B}$ is an interface if there exists $c_- \in (0, c)$, $c_+ \in (c, 1)$ and $j_1 \leqslant j_2 \in \mathbb{R}$ such that $T^{j_2}\varphi_+ \leqslant u \leqslant T^{j_1}\varphi_-$ where $\varphi_+ = c_+H$ and $\varphi_- = c_- + (1 - c_-)P_\ell H$. By monotony, we have for every $t \in \mathbb{N}$

$$T^{j_2}F^t\varphi_+ \leqslant F^t u \leqslant T^{j_1}F^t\varphi_-. \tag{2}$$

Since $F^t\varphi_+(+\infty) = f^t(c_+)$ and $F^t\varphi_-(-\infty) = f^t(c_-)$, it follows that $F^t u$ is an interface for every t.

In order to prove that the velocity of every interfacial orbit is the front velocity, we first prove the result in the case where f is superstable. We then extend the conclusion to any regular bistable map by using theorem 1.2.

A bistable map is said to be *superstable* if there exists $\delta > 0$ such that $f(x) = 0$ if $x \in [0, \delta]$ and $f(x) = 1$ if $x \in [1 - \delta, 1]$.

Proposition 4.5. *Let h be a distribution function and let f be a superstable map. For every interface u and every $a \in (0, 1)$, we have*

$$\lim_{t\to\infty} \frac{J_a(F^t u)}{t} = v(f, h).$$

Proof. Let u be an interface. According to the arguments at the beginning of this section, one can always assume, by considering the function $F^t u$ for t sufficiently large instead of u, that the inequalities (2) hold for all $t \in \mathbb{N}$ with $c_- \leqslant \delta$ and $c_+ \geqslant 1 - \delta$. By definition of J_a, it results that

$$\liminf_{t\to\infty} \frac{J_a(F^t\varphi_-)}{t} \leqslant \liminf_{t\to\infty} \frac{J_a(F^t u)}{t} \leqslant \limsup_{t\to\infty} \frac{J_a(F^t u)}{t} \leqslant \limsup_{t\to\infty} \frac{J_a(F^t\varphi_+)}{t}.$$

Therefore, we only have to prove that

$$\liminf_{t\to\infty} \frac{J_a(F^t\varphi_-)}{t} = \limsup_{t\to\infty} \frac{J_a(F^t\varphi_+)}{t} = v(f, h).$$

From theorem 1.1, let ϕ be a shape of fronts for the map $h*f$. There exist $j_3 \leqslant j_4 \in \mathbb{R}$ such that $T^{j_4}\varphi_+ \leqslant \phi \leqslant T^{j_3}\varphi_-$. These inequalities imply that, for every $a \in (0, 1)$, the following inequalities hold for all $t \in \mathbb{N}$:

$$j_3 + J_a(F^t\varphi_-) \leqslant J_a(F^t\phi) \leqslant j_4 + J_a(F^t\varphi_+).$$

Now, f is superstable and by assumptions on c_- and c_+, we have $F\varphi_- = F\varphi_+ = FH$ and then $J_a(F^t\varphi_-) = J_a(F^t\varphi_+)$ for all $t \geqslant 1$. Together with the relation $J_a(F^t\phi) = v(f, h)t + J_a(\phi)$ (every superstable map is regular), we finally obtain

$$\lim_{t\to\infty} \frac{J_a(F^t\varphi_-)}{t} = \lim_{t\to\infty} \frac{J_a(F^t\varphi_+)}{t} = v(f, h),$$

which is the desired result. □

Proof of theorem 1.3. We prove that, for any interface u, $\limsup_{t\to\infty} J_a(F^t u)/t \leqslant v(f, h)$. The proof that $\liminf_{t\to\infty} J_a(F^t u)/t \geqslant v(f, h)$ is similar and is left to the reader.

The strategy is to construct, for every a, a family $\{f_\varepsilon\}$ of superstable approximations of f. Given $\varepsilon \in (0, \min\{c, 1-c\})$, we consider the superstable map f_ε defined on [0, 1] as follows:

$$f_\varepsilon(x) = \begin{cases} 0 & \text{if } x \in \left[0, \dfrac{\varepsilon}{2}\right], \\ f(2x-\varepsilon) & \text{if } x \in \left[\dfrac{\varepsilon}{2}, \varepsilon\right], \\ f(x) & \text{if } x \in [\varepsilon, 1-\varepsilon], \\ f(1-\varepsilon) & \text{if } x \in [1-\varepsilon, 1]. \end{cases}$$

Let $F_\varepsilon = h * f_\varepsilon$. We have $f_\varepsilon \leqslant f$ and then $J_a(F^t u) \leqslant J_a(F_\varepsilon^t u)$ for all t.

The map f_ε is superstable with fixed points 0 and $f(1-\varepsilon)$. Since the choice of the fixed points of the local map is arbitrary in our analysis, theorem 1.1 and proposition 4.5 hold for the mappings F_ε. It follows that

$$\lim_{t\to\infty} \frac{J_a(F_\varepsilon^t u)}{t} = v(f_\varepsilon, h)$$

and then

$$\limsup_{t\to\infty} \frac{J_a(F^t u)}{t} \leqslant v(f_\varepsilon, h).$$

From the definition of f_ε, we have $\lim_{\varepsilon\to 0} f_\varepsilon = f$. From theorem 1.2 and comments following this statement, it follows that $\lim_{\varepsilon\to 0} v(f_\varepsilon, h) = v(f, h)$ and hence

$$\limsup_{t\to\infty} \frac{J_a(F^t u)}{t} \leqslant v(f, h). \qquad \square$$

5. Extension to other models

Our results on front dynamics extend to systems that are linear convex combinations of maps of the form $h * f$. This allows us to consider systems where several nonlinearities and several couplings are combined. As shown later, such a combination can be constructed to obtain a lattice dynamical system [2].

5.1. The generalized model

From now on, the dynamical system we are considering is $(\mathcal{B}, F)$, where F is defined by

$$Fu = \sum_{k\in\mathbb{N}} a_k h_k * f_k \circ u, \qquad u \in \mathcal{B}.$$

Here the numbers $a_k \geqslant 0$ and $\sum_{k\in\mathbb{N}} a_k = 1$ (we assume that $a_0 > 0$, which is always possible by shifting the index). The functions h_k are distribution functions and the maps f_k are continuous increasing maps defined on [0, 1] such that there exists $c \in (0, 1)$ so that for every $k \in \mathbb{N}$ we have

$$f_k(x) \leqslant x \quad \text{if } 0 \leqslant x \leqslant c \qquad \text{and} \qquad x \leqslant f_k(x) \quad \text{if } c \leqslant x \leqslant 1.$$

Moreover, we assume that the map

$$f = \sum_{k\in\mathbb{N}} a_k f_k$$

is bistable. Its unstable fixed point is then c.

In addition, we say that the map F is regular if there exists $\delta > 0$ such that for every $k \in \mathbb{N}$ we have

$$|f_k(x) - f_k(y)| \leqslant |x-y| \qquad \text{if } x, y \in (0, \delta) \quad \text{or} \quad \text{if } x, y \in (1-\delta, 1).$$

Example: Lattice dynamical system. Let $\varepsilon \in (0, 1)$ and f be a regular bistable map such that the map f_0 defined on $[0, 1]$ by $f_0(x) = (f(x) - \varepsilon x)/(1 - \varepsilon)$ is increasing. Let $f_1(x) = x$ for all x, $a_0 = 1 - \varepsilon$, $a_1 = \varepsilon$, $a_k = 0$ if $k > 1$, $h_0 = H$ and $h_1 = \frac{1}{2}(T^1 H + T^{-1} H)$.

The map

$$Fu(x) = \sum_{k\in\mathbb{N}} a_k h_k * f_k \circ u(x) = f \circ u(x) + \frac{\varepsilon}{2}(u(x-1) - 2u(x) + u(x+1))$$

satisfies the desired properties and is regular.

5.2. *Dynamics of fronts*

The map F keeps $\mathcal{B}$ invariant; so the dynamics is well-defined. Moreover, it shares the following properties with the previous model (where $a_k = \delta_{k,0}$).

- Monotony (respectively s-homogeneity) by monotony (respectively s-homogeneity) of $h_k * f_k$, the non-negativity of the a_k and the uniform convergence of the series.
- If u is a constant function, then Fu is also constant and $Fu = f \circ u$.

In particular, these properties imply that for every $u \in \mathcal{I}$, we have $Fu(\pm\infty) = f(u(\pm\infty))$.

Therefore, every statement that only uses these properties and not the explicit expression of the mapping also holds in the present model. Modifying the other statements so that they also hold, we obtain the following extension of the results on the dynamics of fronts and interfacial orbits.

Theorem 5.1.

(1) The map $F = \sum_{k\in\mathbb{N}} a_k h_k * f_k$ has fronts of velocity v for some $v \in \mathbb{R}$. In addition, if F is regular, then this velocity is unique and is denoted by $v(F)$.

(2) Let $\{F_n\}_{n\in\mathbb{N}}$ be a sequence of maps defined by $F_n = \sum_{k\in\mathbb{N}} a_{k,n} h_{k,n} * f_{k,n}$, where for every n, the assumptions of section 5.1 hold, where for every k, $\lim_{n\to\infty} f_{k,n} = f_k$ and $\lim_{n\to\infty} d(h_{k,n}, h_k) = 0$ and where $\lim_{n\to\infty} \sum_{k\in\mathbb{N}} |a_{k,n} - a_k| = 0$. Assume that the maps F and F_n are regular. Then, we have $\lim_{n\to\infty} v(F_n) = v(F)$.

(3) If F is regular, then for every interface u and every $a \in (0, 1)$, we have $\lim_{t\to\infty} J_a(F^t u)/t = v(F)$.

These results on front dynamics can be extended to other models. For instance, if g is a homoeomorphism of $[0, 1]$, then by a simple change of variable, the conclusions of theorem 5.1 also hold for the map

$$u \mapsto g^{-1}\Big(\sum_{k\in\mathbb{N}} a_k h_k * f_k \circ g \circ u\Big), \qquad u \in \mathcal{B},$$

which can be viewed as a nonlinear coupling between the local maps $f_k \circ g$.

5.3. *Analysis of the dynamics*

The results claimed in theorem 5.1 follow from an analysis similar to the one developed for the previous model. There are two essential steps: the first one, reported in this section, deals with sub-front properties (similar to section 2) and various constructions of fronts (similar to section 3). The second step is about the proof of the theorem and will be presented in the next section.

First, one shows that, if the supremum of sub-front velocities, namely $\bar{v}$, is finite, then there exists a corresponding minimal sub-front, namely $\eta_{\bar{v},c_+}$. More precisely, lemmas 2.1

and 2.2 can be easily repeated for the present map since their proofs do not use the explicit expression of F but only the properties mentioned earlier.

Second, by investigating the properties of sub-fronts between c and 1 (for which the supremum velocity has been denoted as $\bar{v}_s$) and the properties of super-fronts between 0 and c (with corresponding infimum velocity $\bar{v}_i$), one obtains the following statement analogous to lemma 2.3.

Lemma 5.2. *If the distribution function* $h = \sum_{k\in\mathbb{N}} a_k h_k$ *is not degenerate, then we have* $\bar{v}_s < \bar{v}_i$.

The proof is very similar to that of lemma 2.3 and relies on the property that all maps f_k cross the diagonal at the same point c. Note that if h is degenerate, then all h_k are equal and we have $F = h * f$, for which the existence of fronts has been proved.

Given sub-front properties, one can proceed with the construction of fronts. Recall that this construction was threefold in section 3. First, we developed a general construction by means of theorem 3.1. Second, a sufficient condition to apply this theorem was obtained (proposition 3.3) and we finally proved an approximation statement (theorem 3.5).

Theorem 3.1 claims that the condition $\bar{v}_s < \bar{v}$ implies the existence of fronts. For the present map, this theorem still holds since its proof only uses monotony and s-homogeneity of the map, the sub-front $\eta_{\bar{v},c_+}$ and the definitions of $\bar{v}$ and $\bar{v}_s$, but not the explicit expression of F.

Proposition 3.3 becomes the following statement, which together with theorem 3.1, indicates that the conditions $\bar{f}'_{0_r}(c) = +\infty$ and $v_{\min}(h_0) = -\infty$ imply the existence of fronts (because we always have $\bar{v} > -\infty$).

Proposition 5.3. *If* $\bar{f}'_{0_r}(c) = +\infty$ *and if* $v_{\min}(h_0) = -\infty$, *then* $\bar{v}_s = -\infty$.

As before, this result is an immediate consequence of the following statement.

Lemma 5.4. *Given* $c_+ \in (c, 1)$, *consider the function* $\psi_{c_+} = c + (c_+ - c)P_\ell H$. *We have*

(1) $\bar{v}_s \leqslant J_{c_+}(F\psi_{c_+})$,
(2) $P_\ell h_0(J_{c_+}(F\psi_{c_+})) \leqslant (c_+ - c)/(a_0(f_0(c_+) - c))$.

The approximation technique is based on the following statement (analogous to theorem 3.5).

Theorem 5.5. *Let* $\{F_n\}_{n\in\mathbb{N}}$ *be a sequence of maps defined by* $F_n = \sum_{k\in\mathbb{N}} a_{k,n} h_{k,n} * f_{k,n}$, *where for every n, the assumptions of section 5.1 hold, where for every k,* $\lim_{n\to\infty} f_{k,n} = f_k$ *and* $\lim_{n\to\infty} d(h_{k,n}, h_k) = 0$ *and where* $\lim_{n\to\infty} \sum_{k\in\mathbb{N}} |a_{k,n} - a_k| = 0$. *Assume that* $F = \sum_{k\in\mathbb{N}} a_k h_k * f_k$ *satisfies also the assumptions of section 5.1, where* $h = \sum_{k\in\mathbb{N}} a_k h_k$ *is not degenerate.*

If each map F_n *has fronts of velocity* v_n *and if* $\lim_{n\to\infty} v_n = v$, *then the map F has fronts of velocity* v.

Proof. The proof resembles that of theorem 3.5. Let $\phi_\infty = \lim_{n\to\infty} T^{-j_n}\phi_n$ and $\phi = P_r\phi_\infty$, where ϕ_n is a shape of front of velocity v_n of F_n and $j_n = J_{c_-}(\phi_n)$ (respectively $j_n = J_{c_+}(\phi_n)$) if one assumes $v \leqslant \bar{v}_s$ (respectively $v > \bar{v}_s$). We prove that

$$\lim_{n\to\infty} F_n T^{-j_n}\phi_n(x) = F\phi_\infty(x)$$

at all points where $F\phi_\infty$ is continuous.

As before, for every $k \in \mathbb{N}$, we have $\lim_{n\to\infty} h_{k,n} * f_{k,n} \circ T^{-j_n}\phi_n(x) = h_k * f_k \circ \phi_\infty(x)$ for every x such that $h_k * f_k \circ \phi_\infty$ is continuous.

Given $\varepsilon > 0$, let p_ε be such that $\sum_{k>p_\varepsilon} a_k < \varepsilon/4$. Let $x \in \mathbb{R}$ be a point where $F\phi_\infty$ is continuous. Every function $h_k * f_k \circ \phi_\infty$ is continuous at x. Indeed, if $h_k * f_k \circ \phi_\infty$

is discontinuous at x for some k, then monotony implies that $F\phi_\infty$ is also discontinuous at this point. Then there exists $n_\varepsilon \in \mathbb{N}$ such that for all $n > n_\varepsilon$, we have for every $k \in \{0, \ldots, p_\varepsilon\}$

$$|h_{k,n} * f_{k,n} \circ T^{-j_n}\phi_n(x) - h_k * f_k \circ \phi_\infty(x)| < \frac{\varepsilon}{4} \qquad \text{and} \qquad \sum_{k\in\mathbb{N}} |a_{k,n} - a_k| < \frac{\varepsilon}{4}.$$

Therefore, for every $n > n_\varepsilon$, we have

$$\begin{aligned}|F_n T^{-j_n}\phi_n(x) - F\phi_\infty(x)| &\leqslant 2\sum_{k>p_\varepsilon} a_k + \sum_{k\in\mathbb{N}} |a_{k,n} - a_k| \\ &\quad + \sum_{k=0}^{p_\varepsilon} a_k |h_{k,n} * f_{k,n} \circ T^{-j_n}\phi_n(x) - h_k * f_k \circ \phi_\infty(x)| \\ &< 3\frac{\varepsilon}{4} + \frac{\varepsilon}{4}\sum_{k=0}^{p_\varepsilon} a_k \leqslant \varepsilon,\end{aligned}$$

which shows the desired limit. Applying P_r, it follows that $F\phi = T^v\phi$.

In order to conclude the proof, it remains to be shown that ϕ has the appropriate asymptotic behaviour. This can be done in a way similar to that of the proof of theorem 3.5 by considering separately the cases $v \leqslant \bar{v}_s$ and $v > \bar{v}_s$. □

5.4. Sketch of proof of theorem 5.1

As in the previous model, the proof of theorem 5.1 uses the following auxiliary result analogous to lemma 4.1.

Lemma 5.6. *Let $c_- \in (0, c)$ and $c_+ \in (c, 1)$. The following inequalities hold:*

$$c_- - f(c_-) \leqslant \sum_{k\in\mathbb{N}} a_k(1 - f_k(c_-))h_k(\bar{v}) \qquad \textit{and} \qquad \sum_{k\in\mathbb{N}} a_k f_k(c_+) P_\ell h_k(\bar{v}) \leqslant c_+.$$

(1) In order to prove the existence of fronts in a way similar to that of section 4.1, we consider the numbers

$$v^- = \inf\left\{v \in \mathbb{R} : c_- - f(c_-) \leqslant \sum_{k\in\mathbb{N}} a_k(1 - f_k(c_-))h_k(v)\right\}$$

$$v^+ = \sup\left\{v \in \mathbb{R} : \sum_{k\in\mathbb{N}} a_k f_k(c_+) P_\ell h_k(v) \leqslant c_+\right\}$$

and, when ε is sufficiently small, the maps f_0^ε and the functions h_0^ε defined by

$$f_0^\varepsilon(x) = \begin{cases} f_0(x) & \text{if } x \in [0, c], \\ \max\{c + \varepsilon\sqrt{x - c}, f_0(x)\} & \text{if } x \in (c, 1], \end{cases}$$

$$h_0^\varepsilon(x) = \begin{cases} \max\{\varepsilon \mathrm{e}^{x-v^-}, h_0(x)\} & \text{if } x < v^-, \\ h_0(x) & \text{if } x \geqslant v^-. \end{cases}$$

We have $\bar{f}_{0_r}^{\varepsilon\prime}(c) = +\infty$ and $v_{\min}(h_0^\varepsilon) = -\infty$ for every ε. By lemma 5.3 and (the generalized) theorem 3.1, each map F^ε defined by $F^\varepsilon = a_0 h_0^\varepsilon * f_0^\varepsilon + \sum_{k\geqslant 1} a_k h_k * f_k$ has fronts of velocity v_ε. Moreover, lemma 5.6 implies that $v^- \leqslant v_\varepsilon \leqslant v^+$.

Furthermore, the maps f_0^ε converge to f_0 and the functions h_0^ε converge to h_0. By applying theorem 5.5, we conclude that F has fronts of velocity $v = \limsup_{\varepsilon\to 0} v_\varepsilon$.

The proof of uniqueness is nearly the same as the one given in section 4.2. Indeed, only the last part of the proof of lemma 4.3 (after relation (1)) needs to be adapted to the present map. This adaptation is left to the reader.

(2) As in the proof of theorem 1.2, this statement follows by using the approximation theorem 5.5 once the velocities $v_n := v(F_n)$ have been shown to be bounded. Here, we only prove that v_n is bounded from below; the proof of the upper bound is left to the reader.

Let $c_- \in (0, c)$ and let $\varepsilon \in (0, c_- - f(c_-))$ be fixed. Let p_ε be such that $\sum_{k>p_\varepsilon} a_{k,n} < \varepsilon/3$ for all n. Let $n_0 \in \mathbb{N}$ be such that for every $n > n_0$, we have $c_- - f(c_-) - \varepsilon/3 \leqslant c_- - f_{\cdot,n}(c_-)$, where $f_{\cdot,n} = \sum_{k\in\mathbb{N}} a_{k,n} f_{k,n}$.

Now, let $n_1 \geqslant n_0$ be such that for all $n > n_1$, we have for every $k \in \{0, \dots, p_\varepsilon\}$

$$h_{k,n}(v_n) \leqslant h_k\left(v_n + \frac{\varepsilon}{6}\right) + \frac{\varepsilon}{6}.$$

Finally, let $n_2 \geqslant n_1$ be such that for all $n > n_2$, we have for every $k \in \{0, \dots, p_\varepsilon\}$, $a_{k,n}(1 - f_{k,n}(c_-)) \leqslant a_k(1 - f_k(c_-)) + \varepsilon/6$. It follows that, for all $n > n_2$, we have

$$a_{k,n}(1 - f_{k,n}(c_-))h_{k,n}(v_n) \leqslant a_k(1 - f_k(c_-))h_k\left(v_n + \frac{\varepsilon}{6}\right) + \frac{\varepsilon}{3}$$

for every $k \in \{0, \dots, p_\varepsilon\}$, and then by lemma 5.6

$$\begin{aligned} c_- f(c_-) - \varepsilon \leqslant c_- - f_{\cdot,n}(c_-) - \frac{2\varepsilon}{3} &\leqslant \sum_{k=0}^{p_\varepsilon} a_{k,n}(1 - f_{k,n}(c_-))h_{k,n}(v_n) - \frac{\varepsilon}{3} \\ &\leqslant \sum_{k=0}^{p_\varepsilon} a_k(1 - f_k(c_-))h_k\left(v_n + \frac{\varepsilon}{6}\right). \end{aligned}$$

Consequently, the sequence $\{v_n\}$ is bounded from below.

(3) In order to obtain the result on the velocity of interfacial orbits, let us first say that the map F is superstable if there exists $\delta > 0$ such that, for every $k \in \mathbb{N}$, $f_k(x) = 0$ if $x \in [0, \delta]$ and $f_k(x) = 1$ if $x \in [1 - \delta, 1]$.

If F is superstable, then by following the proof of proposition 4.5 and since we still have $F\varphi_- = F\varphi_+ = FH$, we conclude that

$$\lim_{t\to\infty} \frac{J_a(F^t u)}{t} = v(F)$$

for every interface u and every $a \in (0, 1)$.

In the general case, we approximate the map F by the superstable maps $F^\varepsilon = \sum_{k\in\mathbb{N}} a_k^\varepsilon h_k * f_k^\varepsilon$, where

$$a_k^\varepsilon = \begin{cases} a_k & \text{if } k \leqslant 1/\varepsilon, \\ 0 & \text{if } k > 1/\varepsilon \end{cases}$$

and f_k^ε are superstable approximations of f_k that satisfy the assumptions of section 5.1 and are such that $f_k^\varepsilon \leqslant f_k$. Then $J_a(F^t u) \leqslant J_a((F^\varepsilon)^t u)$ for all t and as in the proof of theorem 1.3, this implies that, for every interface u, we have

$$\limsup_{t\to\infty} \frac{J_a(F^t u)}{t} \leqslant v(F).$$

The inequality $\liminf_{t\to\infty} J_a(F^t u)/t \geqslant v(F)$ can be obtained similarly.

Appendix A. On the convolution with a distribution function

In the proof of lemma 2.3, we use lemma A.3. In order to prove this statement, we need two preliminary properties.

Lemma A.1. *Let h be a distribution function. Every monotone function $\psi \in \mathcal{B}$ for which $h * \psi = 0$ is the null function.*

Proof. From the s-homogeneity, we have $\psi(-\infty) = h * \psi(-\infty) = 0$. Similarly, we have $\psi(+\infty) = 0$. By monotony, we conclude that ψ is the null function. □

Lemma A.2. *Let $h \neq H$ be a distribution function. Every right continuous (or left continuous) function ψ of bounded variation satisfying $h * \psi = \psi$ is constant.*

Proof. Applying the Fourier–Stieltjes transform to $h * \psi = \psi$, we obtain

$$\int_{\mathbb{R}} \mathrm{e}^{\mathrm{i}tx}\,\mathrm{d}h(x) \int_{\mathbb{R}} \mathrm{e}^{\mathrm{i}tx}\,\mathrm{d}\psi(x) = \int_{\mathbb{R}} \mathrm{e}^{\mathrm{i}tx}\,\mathrm{d}\psi(x), \qquad t \in \mathbb{R}.$$

However the map $t \mapsto \int_{\mathbb{R}} \mathrm{e}^{\mathrm{i}tx}\,\mathrm{d}\psi(x)$ is continuous [13] and, because $h \neq H$, the set

$$\left\{ t \in \mathbb{R} : \int_{\mathbb{R}} \mathrm{e}^{\mathrm{i}tx}\,\mathrm{d}h(x) = 1 \right\}$$

is countable [13]. Consequently, we have $\int_{\mathbb{R}} \mathrm{e}^{\mathrm{i}tx}\,\mathrm{d}\psi(x) = 0$ for all $t \in \mathbb{R}$. The lemma follows from the uniqueness of the Fourier–Stieltjes transform. □

Lemma A.3. *Let $h \neq H$ be a distribution function.*

(1) If there exists an increasing non-constant bounded function ψ such that $h * \psi \leqslant \psi$, then $\lim_{n\to\infty} h^{*n} = 0$.
(2) If there exists an increasing non-constant bounded function ψ such that $\psi \leqslant h * \psi$, then $\lim_{n\to\infty} h^{*n} = 1$.

Proof. (1) By considering the function $(P_r\psi(x) - \psi(-\infty))/(\psi(+\infty) - \psi(-\infty))$ instead of ψ, we can always assume that ψ is a distribution function. Hence, ψ and all functions $h^{*n} * \psi$ are non-negative.

In addition, using $h * \psi \leqslant \psi$, we obtain the property that the sequence $\{h^{*n} * \psi\}$ is decreasing and the following limit exists:

$$\psi_\infty = \lim_{n\to\infty} h^{*n} * \psi.$$

The function ψ_∞ is increasing and right continuous. By s-homogeneity, we have $h*\psi_\infty = \psi_\infty$. Lemma A.2 implies that ψ_∞ is constant. We have $\psi_\infty \leqslant \psi$ and the condition $\psi(-\infty) = 0$ implies that $\psi_\infty = 0$.

We now apply Helly's selection theorem in order to obtain the limit of a pointwise convergent subsequence of $\{h^{*n}\}$. Let $g = \lim_{i\to\infty} h^{*n_i}$ be such a limit. By using s-homogeneity and the commutation of convolutions, which holds for right continuous functions of bounded variation vanishing at $-\infty$, we have

$$\psi * g = \lim_{i\to\infty} \psi * h^{*n_i} = \lim_{i\to\infty} h^{*n_i} * \psi = \psi_\infty = 0.$$

By applying lemma A.1, we conclude that $g = 0$ and the first statement of the lemma follows. The second statement can be proved similarly. □

Our last property serves the proof of uniqueness of the velocity of fronts.

Lemma A.4. *Let $h \neq H$ be a distribution function. Every non-negative function $\psi \in \mathcal{B}$ such that $\psi(-\infty) = \psi(+\infty) = 0$ and $\psi \leqslant h * \psi$ is the null function.*

Proof. We assume that $\|\psi\| > 0$ and we prove that the conditions on ψ impose $h = H$.

For any $x, \alpha \in \mathbb{R}$, we have

$$\int_{(-\infty,\alpha]} \psi(x-y)\,\mathrm{d}h(y) \leqslant h(\alpha) \sup_{t \geqslant x-\alpha} \psi(t)$$

$$\int_{(\alpha,+\infty)} \psi(x-y)\,\mathrm{d}h(y) \leqslant (1-h(\alpha)) \sup_{t \leqslant x-\alpha} \psi(t).$$

Consequently, our function ψ satisfies the inequality

$$\psi(x) \leqslant h * \psi(x) \leqslant h(\alpha) \sup_{t \geqslant x-\alpha} \psi(t) + (1-h(\alpha)) \sup_{t \leqslant x-\alpha} \psi(t) \tag{A1}$$

for every $x, \alpha \in \mathbb{R}$.

The conditions $\psi(-\infty) = 0$ and $\psi(+\infty) = 0$ imply that the quantity

$$x_\infty = \inf\{x \in \mathbb{R} : \sup_{t \leqslant x} \psi(t) = \|\psi\|\}$$

is a real number. Moreover, there exists a sequence $\{x_n\}_{n\in\mathbb{N}}$ of real numbers such that

$$\lim_{n\to\infty} x_n = x_\infty \qquad \text{and} \qquad \lim_{n\to\infty} \psi(x_n) = \|\psi\|.$$

Fix $\varepsilon > 0$ and let $n_\varepsilon \in \mathbb{N}$ be such that $x_n < x_\infty + \varepsilon/2$ for all $n \geqslant n_\varepsilon$. The relation (A1) implies that for all $n \geqslant n_\varepsilon$ we have

$$\begin{aligned}\psi(x_n) &\leqslant h(\varepsilon) \sup_{t \geqslant x_n-\varepsilon} \psi(t) + (1-h(\varepsilon)) \sup_{t \leqslant x_n-\varepsilon} \psi(t)\\ &\leqslant h(\varepsilon)\|\psi\| + (1-h(\varepsilon)) \sup_{t \leqslant x_\infty-\varepsilon/2} \psi(t).\end{aligned}$$

By taking the limit $n \to \infty$, we obtain

$$0 \leqslant (1-h(\varepsilon))\Big(\sup_{t \leqslant x_\infty-\varepsilon/2} \psi(t) - \|\psi\|\Big).$$

The definition of x_∞ forces $\sup_{t\leqslant x_\infty-\varepsilon/2} \psi(t) - \|\psi\| < 0$. Therefore, $h(\varepsilon) = 1$ for all $\varepsilon > 0$.

Similarly, by introducing the number

$$x^\infty = \sup\{x \in \mathbb{R} : \sup_{t \geqslant x} \psi(t) = \|\psi\|\},$$

one proves that $h(-\varepsilon) = 0$ for all $\varepsilon > 0$ and the lemma follows. □

Appendix B. On the convergence of sequences of increasing functions

In all this work, in particular in the proof of the continuity of the front velocity, we use several results about the convergence of sequences of increasing functions. These results are stated and proved in the present section. The first one concerns limits of translations.

Lemma B.1. *Let $\{\alpha_n\}_{n\in\mathbb{N}}$, $\alpha_n \in \mathbb{R}$ and let $\{\psi_n\}_{n\in\mathbb{N}}$, $\psi_n \in \mathcal{I}$ be two sequences such that $\lim_{n\to\infty} \alpha_n = \alpha \in \mathbb{R}$ and $\lim_{n\to\infty} \psi_n = \psi$. If ψ is continuous at $x-\alpha$, then the following limit exists and we have*

$$\lim_{n\to\infty} T^{\alpha_n}\psi_n(x) = T^\alpha\psi(x).$$

Proof. Given $\varepsilon > 0$, let n_ε be such that $-\alpha - \varepsilon < -\alpha_n < -\alpha + \varepsilon$ for all $n > n_\varepsilon$. Monotony implies that for $n > n_\varepsilon$, we have $\psi_n(x - \alpha - \varepsilon) \leqslant \psi_n(x - \alpha_n) \leqslant \psi_n(x - \alpha + \varepsilon)$ for all $x \in \mathbb{R}$. Using the definition of ψ, it follows that

$$\psi(x - \alpha - \varepsilon) \leqslant \liminf_{n\to\infty} \psi_n(x - \alpha_n) \leqslant \limsup_{n\to\infty} \psi_n(x - \alpha_n) \leqslant \psi(x - \alpha + \varepsilon), \qquad x \in \mathbb{R}.$$

Since ε is arbitrary, we conclude that

$$P_\ell\psi(x - \alpha) \leqslant \liminf_{n\to\infty} \psi_n(x - \alpha_n) \leqslant \limsup_{n\to\infty} \psi_n(x - \alpha_n) \leqslant P_r\psi(x - \alpha), \qquad x \in \mathbb{R},$$

from which the statement follows. □

One can extend the conclusion of the previous lemma to any convolution, provided that the functions ψ_n are distribution functions. This is the scope of the following statement.

Proposition B.2. *Let $\{h_n\}_{n\in\mathbb{N}}$ be a sequence of distribution functions such that $\lim_{n\to\infty} d(h_n, h) = 0$, where h is a distribution function and let $\{\psi_n\}_{n\in\mathbb{N}}$ be a sequence of distribution functions such that $\lim_{n\to\infty} \psi_n = \psi$. We have $\lim_{n\to\infty} d(h_n * \psi_n, h * \psi) = 0$.*

The next statement shows that proposition B.2 can alternatively be stated using the usual convergence of distribution functions. This is used in the proof of existence of fronts using approximations.

Lemma B.3. *Let $\{h_n\}_{n\in\mathbb{N}}$ be a sequence of distribution functions and let h be a distribution function. We have $\lim_{n\to\infty} h_n(x) = h(x)$ for all x where h is continuous iff $\lim_{n\to\infty} d(h_n, h) = 0$.*

Proof of proposition B.2. Let $\varepsilon_n = d(h_n, h)$. Using the commutation $h_n * \psi_n = \psi_n * h_n$ and the properties of the convolution, the definition of $d(\cdot, \cdot)$ implies that

$$h * \psi_n(x - \varepsilon_n) - \varepsilon_n \leqslant h_n * \psi_n(x) \leqslant h * \psi_n(x + \varepsilon_n) + \varepsilon_n$$

for all n and all x. By applying lemma B.1 to the previous inequalities, we obtain

$$h * \psi(x) \leqslant \liminf_{n\to\infty} h_n * \psi_n(x) \leqslant \limsup_{n\to\infty} h_n * \psi_n(x) \leqslant h * \psi(x)$$

if $h * \psi$ is continuous at x. From lemma B.3, we obtain the desired conclusion. □

Proof of lemma B.3. By using the definition of the distance $d(\cdot, \cdot)$, it can immediately be shown that the condition $\lim_{n\to\infty} d(h_n, h) = 0$ implies $\lim_{n\to\infty} h_n(x) = h(x)$ at all continuity points of h.

We prove the converse statement by contradiction. Assume the existence of $\varepsilon > 0$ and of a subsequence $\{n_i\}_{i\in\mathbb{N}}$ such that $d(h_{n_i}, h) > \varepsilon$ for every i. The definition of $d(\cdot, \cdot)$ then implies the existence, for each i, of $x_{n_i} \in \mathbb{R}$ such that either $h(x_{n_i} - \varepsilon) - \varepsilon > h_{n_i}(x_{n_i})$ or $h_{n_i}(x_{n_i}) > h(x_{n_i} + \varepsilon) + \varepsilon$. By taking a subsequence if necessary, we can assume that

$$\text{either } h(x_{n_i} - \varepsilon) - \varepsilon > h_{n_i}(x_{n_i}) \quad \text{for all } i \qquad \text{or} \qquad h_{n_i}(x_{n_i}) > h(x_{n_i} + \varepsilon) + \varepsilon \quad \text{for all } i.$$

We assume that the first inequality holds. The other case can be completed similarly. By taking once again a subsequence if necessary, we can assume that $\lim_{i\to\infty} x_{n_i} = x_\infty$ where $x_\infty \in \mathbb{R}$ or $x_\infty = -\infty$ or $x_\infty = +\infty$.

Assume that $x_\infty \in \mathbb{R}$ and let $x \in (x_\infty - \varepsilon, x_\infty)$ be a point where h is continuous (these points are dense by monotony of h). Then for all i sufficiently large, we have

$$h(x) - \varepsilon \geqslant h(x_{n_i} - \varepsilon) - \varepsilon > h_{n_i}(x_{n_i}) \geqslant h_{n_i}(x)$$

and then, by taking the limit $i \to \infty$, we obtain $h(x) - \varepsilon \geqslant h(x)$, which is in contradiction with the assumption $\varepsilon > 0$.

If $x_\infty = +\infty$, then for every x where h is continuous, there exists i_x such that for every $i > i_x$, we have $1 - \varepsilon \geqslant h(x_{n_i} - \varepsilon) - \varepsilon > h_{n_i}(x_{n_i}) \geqslant h_{n_i}(x)$. By taking the limit $i \to \infty$ and then $x \to +\infty$, we obtain $1 - \varepsilon \geqslant 1$, which is a contradiction.

If $x_\infty = -\infty$, we have $h(x_{n_i} - \varepsilon) - \varepsilon > h_{n_i}(x_{n_i}) \geqslant 0$ and by taking the limit $i \to \infty$, we obtain $-\varepsilon \geqslant 0$, which is also a contradiction. □

Acknowledgments

BF thanks the Departamento de Matemática of the IST and the Grupo da Física-Matemática of Lisbon University for their hospitality and support during this work. We are grateful to the referees, to Ch Chandre and to R Lima for their comments and suggestions, which allowed us to improve the exposition.

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