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Spatially Extended Circle Maps: Monotone Periodic Dynamics of Functions with Linear Growth

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Summary. We introduce and study monotone periodic mappings acting on real functions with linear growth. These mappings represent the nonlinear dynamics of extended systems governed by a diffusive interaction and a periodic potential. They can be viewed as infinite-dimensional analogues of lifts of circle maps. Our results concern the existence and uniqueness of a rotation number and the existence of travelling waves. Moreover, we prove that the rotation number depends continuously on the mapping and we obtain a symmetry condition for this number to vanish. The results are applied to two classes of examples in population dynamics and in condensed matter physics.

1. Spatially Extended Circle Maps

In some situations, modelling nonlinear dynamics of spatially extended systems requires real functions with linear growth. This is the case in condensed matter physics where one-dimensional chains of particles coupled with springs and placed in a periodic potential are represented by doubly infinite real sequences $\{u_s\}_{s\in\mathbb{Z}}$ (u_s represents the location of the s-th particle). In the dissipative limit, the dynamics of such chains, when driven by a constant force, is described by the gradient of a Frenkel-Kontorova (FK) functional [11]. This means that the sequences evolve according to the differential equation

$$\partial_t u_s = V'(u_s) + D + (u_{s-1} - 2u_s + u_{s+1}), \tag{1}$$

where the potential V is periodic $V(x+1) = V(x)$ and $D \in \mathbb{R}$ is the driving force.

The flow associated with this equation has the following main properties:

- It is monotone, i.e., if $\{u_s(0)\} \leq \{v_s(0)\}$, then $\{u_s(t)\} \leq \{v_s(t)\}$ for every $t > 0$. The notation $\{u_s\} \leq \{v_s\}$ means that $u_s \leq v_s$ for all s.
- It is periodic, i.e., if $v_s(0) = u_s(0) + 1$ for all s, then $v_s(t) = u_s(t) + 1$ for all s and all $t > 0$.
- It commutes with space translations, i.e., if $v_s(0) = u_{s-1}(0)$ for all s, then $v_s(t) = u_{s-1}(t)$ for all s and all $t > 0$.
- It is continuous, i.e., if $v_s(0) = \lim_{n\to\infty} u_{s,n}(0)$ for all s, then $v_s(t) = \lim_{n\to\infty} u_{s,n}(t)$ for all s and all $t > 0$.

These properties were used intensively to obtain mathematical results on the existence and stability of time-periodic solutions which are periodic or quasiperiodic along the lattice; see [2] and references therein. Alternatively, these solutions can be viewed as travelling waves whose shape is a periodic function of the real variable. In the particular case without constant force ($D = 0$), the analysis is known as Aubry-Mather theory, see e.g. [3] or [12]. Aubry-Mather theory proves the existence of travelling waves with zero velocity (stationary solutions), whose shape has an arbitrary linear growth rate (mean spacing).

The aim of this article is to prove that the phenomenology of systems satisfying the four properties mentioned above does not depend on the details of the model nor on its specific form. We show that the existence of travelling waves, their stability, and other properties of their dynamics hold for any system satisfying these properties. This concerns systems with discrete coupling involving finitely many sites, as in equation (1), or countably many sites. It also concerns systems with continuous integral coupling as those representing reaction-diffusion systems or neural networks [5].

In order to include discrete and continuous couplings in a unified framework, we consider dynamics of functions of the real variable which can be continuous or discontinuous. As shown in examples of coupled maps, this framework also includes systems where the coupling is partly discrete and partly continuous.

For the sake of simplicity, we work with discrete time dynamical systems. We introduce and study mappings acting on real functions. For these mappings, the previous properties of monotony, periodicity, and continuity are the analogues of the fundamental properties of lifts of maps of the circle, namely monotony, degree 1, and continuity. For this reason, we called our systems *(spatially) extended circle maps*. Despite this analogy, there is an important difference between lifts of circle maps and spatially extended circle maps. The former act in a unique set: $\mathbb{R}$. The latter act in a family of sets parametrised by the mean spacing.

By analysing the dynamics of extended circle maps, we have obtained the following results:

- existence and uniqueness of the rotation number (Proposition 2.1),
- continuous dependence of the rotation number on the system parameters (Proposition 3.2),
- symmetry condition for the rotation number to vanish, relation (9),
- existence of travelling waves (Theorem 2.2).

On one hand, these results are the analogues of the results obtained for lifts of circle maps (see e.g. Chapter 11 in [14] for a review of results for lifts of circle maps). In particular, a function solving the travelling wave equation as in Theorem 2.2 is the analogue of a semiconjugacy to the rotation. We refer to the first part of Section 4 for related results.

On the other hand, these results are the analogues of the results obtained for the equation (1). In particular, the rotation number is the analogue of the average speed defined in [11]. And again, travelling waves are the analogue of periodic orbits.

Finally, in addition to the analysis of general extended circle maps, we consider two classes of examples: lifts of circle maps coupled by convolutions with distribution functions (Section 1.2.1) and discrete time dynamics of FK models (Section 1.2.2). Coupled maps model various reaction-diffusion dynamics, and examples in the second class can be viewed as the discrete time analogue of the flow associated with equation (1). For these examples, we obtain conditions on the components and on the parameters under which the previously mentioned conditions hold. In particular, for each class, we translate the assumptions for continuous dependence of the rotation number and the assumptions for the rotation number to vanish.

In order to conclude this section, we point out that most results in the paper can be expressed in the framework of wave theory. Indeed, the travelling waves, as described in Section 2.2, can be written $\psi(\alpha x - \nu_\alpha t)$ where the shape of the wave ψ is an increasing function such that $\psi(x+1) = \psi(x) + 1$. In this framework, the mean spacing α is a wave number and the rotation number ν_α is a frequency.

Consequently, the result on the existence of a unique rotation number for every mean spacing (Proposition 2.1) shows that a dispersion relation $\alpha \mapsto \nu_\alpha$ can be established for extended circle maps. The result on the existence of travelling waves (Theorem 2.2) proves that, given any number $\alpha \in \mathbb{R}$, the number ν_α is actually a frequency of a travelling wave with wave number α.

Moreover, it is shown that the dispersion relation is a continuous relation (Proposition 3.1) that continuously depends on the parameters of the system (Proposition 3.2). In addition, a symmetry condition is obtained for the dispersion condition to be trivial, i.e., for the existence of stationary waves (equation (9)).

1.1. Definitions

Discrete time dynamical systems are characterised by a set (the phase space) and a mapping in this set. Inspired by sequences with bounded width (Birkhoff sequences) [1], [2], [12], our phase space consists of functions at bounded distance from the function $x \mapsto \alpha x + \beta$ where α is fixed. Precisely, if $\mathcal{B}$ denotes the set of Borel measurable functions from $\mathbb{R}$ into itself, given $\alpha \in \mathbb{R}$, let

$$\mathcal{M}_\alpha = \{u \in \mathcal{B} \ : \ \exists c \geq 0 : \ \forall x \in \mathbb{R} \ |u(x) - \alpha x| \leq c\}.$$

The set $\mathcal{M}_\alpha$ is called the set of functions with mean spacing α and bounded width. In $\mathcal{M}_\alpha$, the dynamics is generated by mappings defined as follows.

Definition 1.1. Given $\alpha \in \mathbb{R}$, a (spatially) extended circle map is a mapping $F\colon \mathcal{M}_\alpha \to \mathcal{B}$ with the following properties:

- F is increasing, i.e., if $u \leq v$, then $Fu \leq Fv$, where $u \leq v$ means $u(x) \leq v(x)$ for all $x \in \mathbb{R}$,
- F is periodic, i.e., $F(u+1) = Fu + 1$, where $(u+1)(x) = u(x) + 1$ for all $x \in \mathbb{R}$,
- F is homogeneous, i.e., $FT^{\nu} = T^{\nu}F$, where given $\nu \in \mathbb{R}$, we have $T^{\nu}u(x) = u(x-\nu)$ for all $x \in \mathbb{R}$,
- F is continuous, i.e., if $\lim\limits_{n\to\infty} u_n = u$, then $\lim\limits_{n\to\infty} Fu_n = Fu$, where $\lim\limits_{n\to\infty} u_n = u$ means $\lim\limits_{n\to\infty} u_n(x) = u(x)$ for all $x \in \mathbb{R}$.

In Section 4, spatially extended circle maps with different properties of periodicity and homogeneity are presented. However, their analysis turns out to be similar to the present one.

As we said in the introduction, periodicity is the analogue of a lift of a circle map of degree 1. We use "periodicity" because an extended circle map does not need to be restricted to a "lift" of some map on a compact manifold (see e.g. Chapter 8 in [14] for a survey of the degree).

The condition $\sup\limits_{x\in\mathbb{R}} |u(x) - \alpha x| \leq c$ implies the following inequalities:

$$\lfloor \alpha y - 2c \rfloor \leq T^{-y}u - u \leq \lceil \alpha y + 2c \rceil, \quad \forall y \in \mathbb{R}, \tag{2}$$

where $\lfloor x \rfloor$ is the largest integer not larger than x and $\lceil x \rceil$ is the smallest integer not smaller than x. By using monotony, periodicity, and homogeneity, it is straightforward to show that Fu also satisfies (2). Since any function u satisfying (2) satisfies the inequalities

$$-2c - 1 + u(0) \leq u(x) - \alpha x \leq 2c + 1 + u(0), \quad \forall x \in \mathbb{R},$$

it follows that $F(\mathcal{M}_\alpha) \subset \mathcal{M}_\alpha$, and the dynamics is well-defined.

It may occur, as in the examples presented below, that an extended circle map will be defined in the family $\{\mathcal{M}_\alpha\}_\alpha$ where α runs over a bounded interval or over the whole $\mathbb{R}$. In addition, an extended circle map may be defined on a bigger set, as in Section 3.1.1. In these cases, each $\mathcal{M}_\alpha$ is regarded as an invariant subset of the larger phase space. In order to extend the scope of the paper, we shall also consider as extended circle maps those mappings defined on (invariant) subsets of $\mathcal{M}_\alpha$.

1.2. Examples

1.2.1. Coupled Maps. Let f be a lift of a circle map (i.e., f is an increasing, continuous map from $\mathbb{R}$ into itself for which $f(x+1) = f(x) + 1$ for all x) and let h be a distribution function, i.e., a right continuous increasing function on $\mathbb{R}$ with $h(-\infty) = 0$ and $h(+\infty) = 1$; see e.g. [17]. Assume that the Lebesgue-Stieltjes integral $\int_{\mathbb{R}} x dh(x)$ exists (see Appendix B for an equivalent condition), and consider the mapping F defined by

$$Fu(x) = \int_{\mathbb{R}} (f \circ u)(x-y) dh(y), \quad \forall x \in \mathbb{R}. \tag{3}$$

This map is defined on $\mathcal{M}_\alpha$ for every $\alpha \in \mathbb{R}$. Indeed, for every $u \in \mathcal{M}_\alpha$, we have $f \circ u \in \mathcal{M}_\alpha$ and the assumption $|\int_{\mathbb{R}} x dh(x)| < +\infty$ implies that the integral in (3) exists for all x.

Moreover, the fact that the local map f is a circle map implies that the mapping $u \mapsto f \circ u$ is an extended circle map. The fact that the function h is a distribution function implies that the mapping $u \mapsto (h * u)(x) := \int_{\mathbb{R}} u(x-y)dh(y),\ \forall x \in \mathbb{R}$ is an extended circle map [7]. Finally, since the composition of two extended circle maps is an extended circle map, the result is that the coupled map $u \mapsto Fu = h * f \circ u$ defined by (3) is an extended circle map.

In addition, since convex linear combinations of extended circle maps are extended circle maps, mappings of the form $Fu = \sum_k a_k h_k * f_k \circ u$ are extended circle maps provided that $a_k \geq 0$, $\sum_k a_k = 1$, h_k are distribution functions and f_k are lifts of circle maps.

Coupled maps are models with discrete coupling when h is a discrete distribution function, or with continuous coupling when h is continuous. In addition, by considering a distribution function that is a convex combination of a discrete distribution function and a continuous one, one obtains a coupled map for which the coupling is partly discrete and partly continuous. Coupled maps with local maps restricted on a bounded interval have been introduced in [18] as models for population dynamics or population ecology.

In the case where h is a lattice distribution, these systems are known as coupled map lattices [13]. Coupled map lattices were introduced and studied as simple space-time discrete dynamical models in various settings, such as alloy solidification, crystal growth, chemical reactions, and traffic flows.

Although they are defined on lattices, it is sometimes convenient and useful to consider coupled map lattices in sets of real functions. This point of view has been applied to front dynamics in bistable systems. It allowed us to prove that the dependence of the front's velocity on a local map parameter is a Devil's staircase [6].

The coupled maps we consider differ slightly from those studied in [4] and references therein. Indeed, we are coupling lifts of circle maps, whereas these studies deal with coupling circle maps, and thus with the coupling being an operator from $(S^1)^{\mathbb{Z}}$ into itself.

Finally, coupled maps with absolutely continuous coupling function h can be viewed as discrete time analogues of neural networks models, as in [10]. Indeed, replacing $\partial_t u$ by $Fu - u$ in the equation $\partial_t u + u = h * f \circ u$ (equation (2.2) in [10]) makes F a coupled map.

1.2.2. Dissipative Dynamics of Frenkel-Kontorova Models. As it is said in the introduction, FK models represent one-dimensional chains of interacting particles in a periodic potential. The standard model involves only nearest neighbours and is defined by using a generating function. A generating function is a C^2 function $g\colon \mathbb{R}^2 \to \mathbb{R}$ satisfying the periodic condition $g(x+1, y+1) = g(x, y)$ and such that the partial derivative $g''_{12}(x, y) \leq 0$ for all $(x, y) \in \mathbb{R}^2$ (twist condition). Given a generating function g, a number $D \in \mathbb{R}$ representing the constant force, and $\varepsilon > 0$, which can be viewed as a discretization step, consider the map F_ε defined by

$$F_\varepsilon u(x) = u(x) - \varepsilon(g'_2(u(x-1), u(x)) + g'_1(u(x), u(x+1)) + D), \quad \forall x \in \mathbb{R}.$$

This map can be viewed as a discrete time analogue of the dissipative dynamics $\partial u_t(x) = -g'_2(u(x-1), u(x)) - g'_1(u(x), u(x+1)) - D$ investigated in [2], [11]. In particular, fixed points of F_ε correspond to stationary points of the latter.

It is straightforward to show that F_ε is periodic, homogeneous, and continuous for every $\varepsilon > 0$. Moreover, one can show that it is increasing on a subset of $\mathcal{M}_\alpha$ provided that the discretisation step is small enough. To that purpose, given $\alpha \in \mathbb{R}$ and $c \geq 0$, let the set be defined by

$$\mathcal{M}_\alpha^c = \{u \in \mathcal{B} \;:\; \forall y \geq 0,\; \lfloor \alpha y - c \rfloor \leq T^{-y}u - u \leq \lceil \alpha y + c \rceil\}.$$

Every set $\mathcal{M}_\alpha^c$ ($c \geq 0$) is invariant under the action of an extended circle map. (Notice also that $\mathcal{M}_\alpha^c$ remains unchanged if we substitute $\forall y \geq 0$ by $\forall y \in \mathbb{R}$ in the definition.)

Lemma 1.2. *For every $L > 0$, there exists $\varepsilon_L > 0$ such that, for every $\varepsilon \in (0, \varepsilon_L]$, the map F_ε is increasing on $\mathcal{M}_\alpha^c$ for every $|\alpha| \leq L$ and $c \leq L$.*

Proof. Let $u \in \mathcal{M}_\alpha^c$ where $|\alpha| \leq L$ and $c \leq L$. We have $|u(x+1) - u(x)| \leq 2L + 1$ and $|u(x-1) - u(x)| \leq 2L + 1$ for all $x \in \mathbb{R}$. Let $M_L < +\infty$ be the supremum of $|g_{22}''(x, y) + g_{11}''(y, z)|$ when $y \in [0, 1]$ and $|x - y|, |y - z| \leq 2L + 1$. By periodicity of g, M_L is an upper bound for the supremum of $|g_{22}''(u(x-1), u(x)) + g_{11}''(u(x), u(x+1)|$ over $u \in \mathcal{M}_\alpha^c$. Let $\varepsilon_L = \frac{1}{M_L}$ (ε_L is arbitrary if $M_L = 0$).

Given $\varepsilon \in (0, \varepsilon_L]$, let $G(x, y, z) = y - \varepsilon(g_2'(x, y) + g_1'(y, z) + D)$. The twist condition implies that $\frac{\partial G(x,y,z)}{\partial x} \geq 0$ and $\frac{\partial G(x,y,z)}{\partial z} \geq 0$ for every $(x, y, z) \in \mathbb{R}^3$. Moreover, for every (x, y, z) such that $|x - y| \leq 2L + 1$ and $|y - z| \leq 2L + 1$, we have

$$\frac{\partial G(x, y, z)}{\partial y} = 1 - \varepsilon(g_{22}''(x, y) + g_{11}''(y, z)) \geq 1 - \varepsilon M_L \geq 0.$$

Consequently, given $u, v \in \mathcal{M}_\alpha^c$ such that $u \leq v$, we have

$$F_\varepsilon u \;=\; G(T^1 u, u, T^{-1}u) \leq G(T^1 v, v, T^{-1}v) = F_\varepsilon v. \qquad \square$$

Similar arguments can be applied to FK models with interactions involving more neighbours than only the nearest ones. For instance, one can consider the map inspired by [1],

$$F_\varepsilon u(x) = u(x) - \varepsilon \Delta(u(x - l), \ldots, u(x + m)),$$

where Δ: $\mathbb{R}^{m+l+1} \to \mathbb{R}$ is a C^2 periodic function such that every partial derivative $\frac{\partial \Delta(x_{-l},\ldots,x_m)}{\partial x_s} \leq 0$ for every $s \neq 0$. Just as in the previous case, one can show that F_ε is an extended circle map for every set in the family $\{\mathcal{M}_\alpha^c\}_{|\alpha| \leq L, c \leq L}$ provided that ε is small enough.

Although extended circle maps of this section are defined only in $\mathcal{M}_\alpha^c$ rather than in the whole $\mathcal{M}_\alpha$, this restriction does not affect the properties proved in the sequel. Indeed, these properties require only that the mapping is defined in $\mathcal{M}_\alpha^0$ (or in $\mathcal{M}_{\alpha,\alpha'}^0$ in Section 3.1.1), which is an invariant subset of $\mathcal{M}_\alpha^c$ (or of $\mathcal{M}_{\alpha,\alpha'}^c$) for every $c \geq 0$.

2. Rotation Number and Travelling Waves

2.1. Existence and Uniqueness of the Rotation Number

As for the study of lifts of circle maps, the analysis of extended circle maps starts by showing existence and uniqueness of a rotation number, that is to say, of an average velocity of vertical displacements of any function in phase space. We shall also obtain estimates on the horizontal displacements of functions in phase space.

In a second step, travelling waves in the subset $\mathcal{M}_\alpha^0$ of periodic functions will be analysed. (Recall that the sets $\mathcal{M}_\alpha^c$ have been defined by

$$\mathcal{M}_\alpha^c = \{u \in \mathcal{B} \ : \ \forall y \geq 0, \ \lfloor \alpha y - c \rfloor \leq T^{-y}u - u \leq \lceil \alpha y + c \rceil\}.)$$

For $\alpha = 0$, the subset $\mathcal{M}_\alpha^0$ consists of constant functions. However, no constant function can solve the travelling wave equation when the rotation number is not zero. Therefore, the analysis of travelling waves does not apply to the case $\alpha = 0$.

Despite this fact, the case $\alpha = 0$ can be solved independently. Indeed, by homogeneity of F, the image of a constant function is a constant function. In other words, the dynamics in the set of constant functions reduces to that of a real map, say f_F. That is to say, we have $Fu = f_F \circ u$ if u is constant. By monotony, periodicity, and continuity of F, the map f_F is a lift of a circle map. Consequently, the limit $\lim\limits_{t\to\infty} \frac{F^t u(x)}{t} = \lim\limits_{t\to\infty} \frac{f_F^t \circ u(x)}{t}$ exists and does not depend on u for every constant function and hence for every function in $\mathcal{M}_0$ (since every such function is bounded from below and from above by constant functions).

Furthermore, we can assume that $\alpha > 0$ because if $\{u^t\}_{t\in\mathbb{N}}$ is an orbit of F in $\mathcal{M}_\alpha$, then $\{Su^t\}_{t\in\mathbb{N}}$ is an orbit of $S \circ F \circ S$ in $\mathcal{M}_{-\alpha}$ and $S \circ F \circ S$ is an extended circle map. Here $Su(x) = u(-x)$ for all $x \in \mathbb{R}$ is the reflection on the line. (A particular case of symmetric coupling occurs when F commutes with S since then $S \circ F \circ S = F$.)

Horizontal displacements will be measured by using the quantity

$$J(u) = \inf\{x \in \mathbb{R} \ : \ u(x) \geq 0\},$$

which is finite for every $u \in \mathcal{M}_\alpha$ with $\alpha > 0$ and satisfies the properties $J(T^\nu u) = J(u) + \nu$ and $u \leq v$ implies $J(u) \geq J(v)$.

Given $\alpha > 0$, let $\mathcal{N}_\alpha := \mathcal{M}_\alpha^0$. The set $\mathcal{N}_\alpha$ is the subset of increasing functions u such that $T^{-\frac{1}{\alpha}} u = u + 1$.

Proposition 2.1. *Let an extended circle map F and $\alpha > 0$ be fixed. For every $u \in \mathcal{M}_\alpha$ and every $x \in \mathbb{R}$, the limit $\nu_\alpha := \lim\limits_{t\to\infty} \frac{F^t u(x)}{t}$ exists and does not depend on x or u.*

Furthermore, we have $|J(F^t u) + \frac{\nu_\alpha t}{\alpha}| \leq \frac{c+1}{\alpha}$ for all $t \in \mathbb{N}$ where c is such that $|u(x) - \alpha x| \leq c$ for all x.

In particular, the rotation number ν_α can be obtained by using the following relation: $\nu_\alpha = -\alpha \lim_{t\to\infty} \frac{J(F^t u)}{t}$.

Proof. Every function $u \in \mathcal{N}_\alpha$ satisfies the inequalities

$$\varphi_\alpha^- \leq T^{-J(u)} u < \varphi_\alpha^+, \tag{4}$$

where $u < v$ means $u \leq v$ and $u \neq v$ and where the functions φ_α^- and φ_α^+ are defined by $\varphi_\alpha^-(x) = \lceil \alpha x \rceil - 1$ and $\varphi_\alpha^+ = \lfloor \alpha x \rfloor + 1$ for all $x \in \mathbb{R}$.

The quantity $j_t := J(F^t \varphi_\alpha^-)$ is finite for every $t \in \mathbb{N}$. The inequalities (4) imply $\varphi_\alpha^- \leq T^{-j_t} F^t \varphi_\alpha^-$ and $T^{-(j_t - \frac{1}{\alpha})} F^t \varphi_\alpha^+ < \varphi_\alpha^+$ because $J(F^t \varphi_\alpha^+) = j_t - \frac{1}{\alpha}$ for all t. Applying F^s, we obtain

$$F^s \varphi_\alpha^- \leq T^{-j_t} F^{t+s} \varphi_\alpha^- \qquad \text{and} \qquad T^{-(j_t - \frac{1}{\alpha})} F^{t+s} \varphi_\alpha^+ \leq F^s \varphi_\alpha^+,$$

and then $j_s \geq j_{t+s} - j_t$ and $(j_{t+s} - \frac{1}{\alpha}) - (j_t - \frac{1}{\alpha}) \geq j_s - \frac{1}{\alpha}$. The subadditivity of the sequence $\{j_t\}_{t \in \mathbb{N}}$ and the superadditivity of $\{j_t - \frac{1}{\alpha}\}_{t \in \mathbb{N}}$ imply that the following limit exists and is finite: $\lim\limits_{t \to \infty} \frac{j_t}{t} = \inf\limits_{t > 0} \frac{j_t}{t} = \sup\limits_{t > 0} \frac{j_t - \frac{1}{\alpha}}{t}$. We denote this quantity by $-\frac{\nu_\alpha}{\alpha}$. A consequence is the following important inequality:

$$-\frac{\nu_\alpha}{\alpha} t \leq j_t \leq -\frac{\nu_\alpha}{\alpha} t + \frac{1}{\alpha} \qquad \forall \varepsilon \in \mathbb{N}. \tag{5}$$

We are now about to prove existence and uniqueness of the rotation number for the function φ_α^-. By applying the inequalities (4) to the function $F^t \varphi_\alpha^-$, we obtain

$$\varphi_\alpha^- - \lceil j_t \alpha \rceil = T^{\frac{1}{\alpha} \lceil j_t \alpha \rceil} \varphi_\alpha^- \leq F^t \varphi_\alpha^- < T^{\frac{1}{\alpha} \lfloor j_t \alpha \rfloor} \varphi_\alpha^+ = \varphi_\alpha^+ - \lfloor j_t \alpha \rfloor.$$

As a consequence, for every $x \in \mathbb{R}$, we have

$$\nu_\alpha = -\lim_{t \to \infty} \frac{\lfloor j_t \alpha \rfloor}{t} \leq \liminf_{t \to \infty} \frac{F^t \varphi_\alpha^-(x)}{t} \leq \limsup_{t \to \infty} \frac{F^t \varphi_\alpha^-(x)}{t} \leq -\lim_{t \to \infty} \frac{\lceil j_t \alpha \rceil}{t} = \nu_\alpha,$$

and thus the rotation number associated with φ_α^- exists and does not depend on x.

In addition, $\varphi_\alpha^- + 1 \leq \varphi_\alpha^+ \leq \varphi_\alpha^- + 2$ and thus the rotation number also exists and does not depend on x for the function φ_α^+. Finally, let $u \in \mathcal{M}_\alpha$ be such that $|u(x) - \alpha x| \leq c$. Then, we have

$$T^{\frac{c+1}{\alpha}} \varphi_\alpha^+ \leq u \leq T^{-\frac{c+1}{\alpha}} \varphi_\alpha^-,$$

which implies the existence and uniqueness of the rotation number for the function u. Moreover and together with (5), the same inequalities imply the inequality $|J(F^t u) + \frac{\nu_\alpha t}{\alpha}| \leq \frac{c+1}{\alpha}$ for all $t \in \mathbb{N}$. □

This estimate on $J(F^t u)$ will be used in the proof of the existence of travelling waves. Note that using $J(u)$ is not necessary to prove the existence and uniqueness of the rotation number. Indeed, every function in $\mathcal{N}_\alpha$ satisfies the inequality $\varphi_\alpha^- \leq u - \lfloor u(0) \rfloor < \varphi_\alpha^+ + 1$. By using this inequality instead of (4) in a reasoning similar to the one in the first part of the proof of Proposition 2.1, one can prove, without using (4), that the limit $\lim\limits_{t \to \infty} \frac{\lfloor F^t u(0) \rfloor}{t}$ exists and does not depend on $u \in \mathcal{N}_\alpha$, and hence does not depend on $u \in \mathcal{M}_\alpha$ for each $\alpha \in \mathbb{R}$.

2.2. Existence of Travelling Waves

The previous section showed that the mean velocity of horizontal displacements of every function in $\mathcal{M}_\alpha$ is $-\frac{\nu_\alpha}{\alpha}$ asymptotically in time; see Proposition 2.1. In this section, we prove the existence of a function whose image under F is exactly its translation by $-\frac{\nu_\alpha}{\alpha}$. By homogeneity, the subsequent orbit consists of functions which are translations of the original functions: It is a travelling wave.

Theorem 2.2. *For any extended circle map F and any $\alpha > 0$, there exists $\phi \in \mathcal{N}_\alpha$ such that $T^{\frac{\nu_\alpha}{\alpha}} F\phi = \phi$.*

We notice that, in the case where $Fu = f \circ u$ and $\alpha = 1$, this theorem states the existence of a semiconjugacy to the translation by ν_1 for the lift of circle map f. Indeed, it states the existence of a lift of a circle map $\phi \in \mathcal{N}_1$ such that $f \circ \phi = T^{-\nu_1}\phi$.

Proof. The proof is inspired from the proof of existence of fronts in [8]. In the first part, we study subsolutions of the travelling wave equation. Given $\nu \in \mathbb{R}$, we define

$$\mathcal{S}(\nu) = \{u \in \mathcal{N}_\alpha \ : \ T^{\frac{\nu}{\alpha}} Fu \leq u \text{ and } J(u) = 0\}.$$

Lemma 2.3. $\nu_\alpha = \inf\{\nu \in \mathbb{R} \ : \ \mathcal{S}(\nu) \neq \emptyset\}$.

Proof of the Lemma. Given $t \in \mathbb{N}$, let the function φ_t be defined by

$$\varphi_t(x) = \min_{0\leq s<t} \{T^{-\frac{s}{t}(j_t - \frac{1}{\alpha})} F^s \varphi_\alpha^+(x)\}, \quad \forall x \in \mathbb{R},$$

where j_t and φ_α^+ were introduced in the proof of Proposition 2.1. The fact that $\varphi_\alpha^+ \in \mathcal{N}_\alpha$ and the properties of F ensure that $\varphi_t \in \mathcal{N}_\alpha$ for every t. Thus all $J(\varphi_t)$ are finite. Moreover, by monotony of F, we have $T^{-\frac{1}{t}(j_t-\frac{1}{\alpha})} F\varphi_t \leq T^{-\frac{s+1}{t}(j_t-\frac{1}{\alpha})} F^{s+1}\varphi_\alpha^+$ for every $0 \leq s < t$. This implies that

$$T^{-\frac{1}{t}(j_t-\frac{1}{\alpha})} F\varphi_t \leq \min_{1\leq s\leq t} T^{-\frac{s}{t}(j_t-\frac{1}{\alpha})} F^s\varphi_\alpha^+ \leq \min_{0\leq s<t} T^{-\frac{s}{t}(j_t-\frac{1}{\alpha})} F^s\varphi_\alpha^+ = \varphi_t,$$

because $T^{-(j_t-\frac{1}{\alpha})} F^t\varphi_\alpha^+ \leq \varphi_\alpha^+$ as indicated by the right inequality (4). We have shown that the set $\mathcal{S}(-\frac{\alpha}{t}(j_t - \frac{1}{\alpha}))$ is not empty for every $t > 0$. Therefore, we have

$$\nu_\alpha = -\alpha \lim_{t\to\infty} \frac{1}{t}\left(j_t - \frac{1}{\alpha}\right) \geq \inf\{\nu \in \mathbb{R} \ : \ \mathcal{S}(\nu) \neq \emptyset\}.$$

On the other hand, we assume that $u \in \mathcal{S}(\nu) \neq \emptyset$ for some $\nu \in \mathbb{R}$. Then $\varphi_\alpha^- \leq u$ by relation (4) and thus $F^t\varphi_\alpha^- \leq F^t u \leq T^{-\frac{\nu}{\alpha}t}u$, which implies $j_t \geq -\frac{\nu}{\alpha}t$, i.e., $\nu \geq -\alpha\frac{j_t}{t} \geq \nu_\alpha$. Consequently, we have $\inf\{\nu \in \mathbb{R} \ : \ \mathcal{S}(\nu) \neq \emptyset\} \geq \nu_\alpha$. □

In a second step of the proof of the theorem, given $\nu \in \mathbb{R}$, we consider the function

$$\eta_\nu(x) = \inf_{u\in\mathcal{S}(\nu)} u(x), \quad \forall x \in \mathbb{R}.$$

Lemma 2.4. $\mathcal{S}(\nu_\alpha) \neq \emptyset$ *and* $\eta_{\nu_\alpha} \in \mathcal{S}(\nu_\alpha)$.

Proof. Assume that $\mathcal{S}(\nu) \neq \emptyset$ and let us show that $\eta_\nu \in \mathcal{S}(\nu)$. It is straightforward to show that $\eta_\nu \in \mathcal{N}_\alpha$ and $J(\eta_\nu) = 0$. Moreover, $\eta_\nu \leq u$ for every $u \in \mathcal{S}(\nu)$ so $T^{\frac{\nu}{\alpha}} F\eta_\nu \leq T^{\frac{\nu}{\alpha}} Fu \leq u$. Since u is arbitrary, we conclude that $T^{\frac{\nu}{\alpha}} F\eta_\nu \leq \eta_\nu$ and then $\eta_\nu \in \mathcal{S}(\nu)$.

It remains to show that $\mathcal{S}(\nu_\alpha) \neq \emptyset$. Assume that $\mathcal{S}(\nu_1) \neq \emptyset$ for some ν_1 (the existence of such ν_1 is ensured by Lemma 2.3) and let $\nu_2 \geq \nu_1$. We have $T^{\frac{\nu_2}{\alpha}} F\eta_{\nu_1} \leq T^{\frac{\nu_1}{\alpha}} F\eta_{\nu_1} \leq \eta_{\nu_1}$ and hence $\eta_{\nu_1} \in \mathcal{S}(\nu_2)$, which implies $\eta_{\nu_2} \leq \eta_{\nu_1}$. By Lemma 2.3 and by using this monotony, it results that the limit $\eta = \lim_{\nu \to \nu_\alpha^+} \eta_\nu$ exists where the convergence is pointwise. By using the continuity of F, we have $F\eta = \lim_{\nu \to \nu_\alpha^+} F\eta_\nu$. By Lemma A.1 in Appendix A, we conclude that

$$T^{\frac{\nu_\alpha}{\alpha}} F\eta(x) = \lim_{\nu \to \nu_\alpha^+} T^{\frac{\nu}{\alpha}} F\eta_\nu(x),$$

at all points x where $T^{\frac{\nu_\alpha}{\alpha}} F\eta$ is continuous. Since $\eta_\nu \in \mathcal{S}(\nu)$, it follows that $T^{\frac{\nu_\alpha}{\alpha}} F\eta(x) \leq \eta(x)$ for all such points. These points are dense in $\mathbb{R}$.

Let P_ℓ be the projection on left continuous functions, i.e., $P_\ell u(x) = \lim_{y \to x^-} u(y)$ is defined on increasing functions. By the continuity of F, P_ℓ and F commute. By applying P_ℓ to the previous inequality, we conclude that $T^{\frac{\nu_\alpha}{\alpha}} FP_\ell\eta \leq P_\ell\eta$. Finally, $P_\ell\eta \in \mathcal{N}_\alpha$, and thus $T^{-J(P_\ell\eta)} P_\ell\eta \in \mathcal{S}(\nu_\alpha)$. □

In the third part, we consider the sequence $\{T^{n\frac{\nu_\alpha}{\alpha}} F^n \eta_{\nu_\alpha}\}_{n \in \mathbb{N}}$. By monotony and homogeneity, we have

$$T^{(n+1)\frac{\nu_\alpha}{\alpha}} F^{n+1}\eta_{\nu_\alpha} \leq T^{n\frac{\nu_\alpha}{\alpha}} F^n\eta_{\nu_\alpha}, \quad \forall n \in \mathbb{N}.$$

In addition, this decreasing sequence is bounded from below. Indeed the inequality $\varphi_\alpha^- \leq \eta_{\nu_\alpha}$ implies $T^{n\frac{\nu_\alpha}{\alpha}} F^n\varphi_\alpha^- \leq T^{n\frac{\nu_\alpha}{\alpha}} F^n\eta_{\nu_\alpha}$, and then $T^{n\frac{\nu_\alpha}{\alpha}+j_n}\varphi_\alpha^- \leq T^{n\frac{\nu_\alpha}{\alpha}} F^n\eta_{\nu_\alpha}$ by relation (4). This shows, by using (5), that $T^{\frac{1}{\alpha}}\varphi_\alpha^- \leq T^{n\frac{\nu_\alpha}{\alpha}} F^n\eta_{\nu_\alpha}$, and hence that the sequence is bounded from below. Consequently, this sequence converges pointwise to the function $\phi := \lim_{n \to \infty} T^{n\frac{\nu_\alpha}{\alpha}} F^n\eta_{\nu_\alpha}$. By continuity, the limit satisfies $T^{\frac{\nu_\alpha}{\alpha}} F\phi = \phi$. Moreover $\phi \in \mathcal{N}_\alpha$, and the proof of the theorem is complete. □

2.3. Nonuniqueness of Travelling Waves and Stability

2.3.1. Nonuniqueness. Let $\phi \in \mathcal{N}_\alpha$ be a function satisfying the travelling wave equation $T^{\frac{\nu_\alpha}{\alpha}} F\phi = \phi$. By homogeneity, every translated $T^x\phi$ ($x \in \mathbb{R}$) satisfies this equation. By continuity, every left continuous function $T^x P_\ell\phi$ and every right continuous function $T^x P_r\phi$ (where P_r is the projection on right continuous functions) also satisfy the travelling wave equation.

In addition to these solutions, the equation may have other solutions. For instance, consider the extended circle map defined by $Fu = f \circ u$, where f is a lift of a circle map such that $f(x) = x$ if $x \in \mathbb{Z} \cup (\mathbb{Z} + \{\frac{1}{2}\})$. Then the functions $x \mapsto \lfloor \alpha x \rfloor$ and $x \mapsto \lfloor \alpha x \rfloor + \frac{1}{2}$ both are solutions of $Fu = u$ in $\mathcal{N}_\alpha$.

The possible existence of solutions not identifiable by translations or by projections suggests that we need to reconsider the whole construction in the proof of Theorem 2.2.

Indeed, the construction can be completed by using the quantity

$$J_\sigma(u) = \inf\{x \in \mathbb{R} \;:\; u(x) \geq \sigma\},$$

instead of $J(u)$. That is to say, one first proves the existence of minimal subsolutions η_σ with $J_\sigma(\eta_\sigma) = 0$ for every σ (which can be restricted to $[0, 1[$ by periodicity). In a second step, one applies $T^{\frac{v_\alpha}{\alpha}} F$ inductively and takes the limit to obtain a family $\{\phi_\sigma\}_{\sigma\in[0,1[}$ of solutions.

This family is well-ordered since the condition $\sigma < \sigma'$ implies $\phi_\sigma \leq \phi_{\sigma'}$. This follows from the fact that $\sigma < \sigma'$ implies $\eta_\sigma \leq \eta_{\sigma'}$ (which is itself a consequence of the inequality $\eta_\sigma \leq T^\gamma \eta_{\sigma'}$ where $\gamma > 0$ is such that $J_\sigma(T^\gamma \eta_{\sigma'}) = 0$). Using continuity, it results that the limits $\phi_\sigma^- := \lim_{\sigma'\to\sigma^-} \phi_{\sigma'}$ and $\phi_\sigma^+ := \lim_{\sigma'\to\sigma^+} \phi_{\sigma'}$ are also solutions which satisfy $\phi_\sigma^- \leq \phi_\sigma \leq \phi_\sigma^+$. It may happen that $\phi_\sigma^- \neq \phi_\sigma$ and/or $\phi_\sigma \neq \phi_\sigma^+$.

Finally, we notice that there may be functions in $\mathcal{N}_\alpha$ that are not solutions of the travelling wave equation but that are solutions of some iterated travelling wave equation. We mean functions ϕ such that $T^{n\frac{v_\alpha}{\alpha}} F^n \phi = \phi$ for some $n > 1$ but not for $n = 1$. For instance, this is the case for the map

$$Fu = \frac{1}{2}(T^1 f \circ u + T^{-1} f \circ u),$$

where f is a lift of a circle map such that every integer is a fixed point. Then $F\phi_1 = \phi_2$ and $F\phi_2 = \phi_1$, where $\phi_1(x) = 2\lfloor \frac{x}{2} \rfloor$ and $\phi_2(x) = 2\lfloor \frac{x-1}{2} \rfloor + 1$.

2.3.2. Stability. We consider the functions ϕ_σ^-, ϕ_σ, and ϕ_σ^+ previously defined. If $\phi_\sigma < \phi_\sigma^+$ (a similar analysis applies if $\phi_\sigma^- < \phi_\sigma$) and if there are no solutions of the travelling wave equation in $\mathcal{N}_\alpha$ lying in the interval $[\phi_\sigma, \phi_\sigma^+]$, then one may wonder about the dynamics of functions that belong to this interval. Their dynamics turns out to be entirely known. Indeed, by Proposition 1 in [9] (whose conditions are checked below), there is a monotone orbit of $T^{\frac{v_\alpha}{\alpha}} F$ connecting ϕ_σ and ϕ_σ^+. (Strictly speaking, Proposition 1 in [9] does not apply here because it is stated in Banach spaces. However, this statement can be extended to locally convex spaces for which fixed point index theory still applies.)

This is to say, there exists a sequence $\{u_t\}_{t\in\mathbb{Z}}$ of functions in $\mathcal{N}_\alpha$ such that $u_{t+1} = T^{\frac{v_\alpha}{\alpha}} F u_t$ for all t and so that

either $u_{t+1} < u_t$, $\lim_{t\to-\infty} u_t = \phi_\sigma^+$ and $\lim_{t\to+\infty} u_t = \phi_\sigma$,
or $u_t < u_{t+1}$, $\lim_{t\to-\infty} u_t = \phi_\sigma$ and $\lim_{t\to+\infty} u_t = \phi_\sigma^+$.

In the first case, ϕ_σ is stable from above since, by monotony, every function in $\mathcal{N}_\alpha$ lying strictly between ϕ_σ and ϕ_σ^+ asymptotically converges to ϕ_σ under the action of $T^{\frac{v_\alpha}{\alpha}} F$. It is unstable from above and ϕ_σ^+ is stable from below in the second case.

In order to apply Proposition 1 in [9], one needs to endow $\mathcal{N}_\alpha$ with a metric such that $T^{\frac{v_\alpha}{\alpha}} F$ is continuous in the subsequent topology and such that every interval $[u, v] \subset \mathcal{N}_\alpha$, where $u < v \in \mathcal{N}_\alpha$, is relatively compact.

The desired topology is the Hausdorff topology, the topology of convergence almost everywhere (see e.g. Chapter 6 in [16]). A compatible distance is defined by

$$d(u, v) = \inf\{\varepsilon > 0 \;:\; T^\varepsilon v - \varepsilon \leq u \leq T^{-\varepsilon} v + \varepsilon\},$$

where $u, v \in \mathcal{N}_\alpha$. Actually, $d(\cdot, \cdot)$ is a distance in the quotient set $\mathcal{N}_\alpha / \sim$, where "$\sim$" means equality almost everywhere. Precisely, it can be shown that $\lim\limits_{n\to\infty} d(u_n, u) = 0$ iff $\lim\limits_{n\to\infty} u_n(x) = u(x)$ for every x where u is continuous. A similar proof is given in [8] in the case where u_n and u are distribution functions.

Helly's selection theorem [15] implies that every interval $[u, v] \subset \mathcal{N}_\alpha$ is relatively compact in the Hausdorff topology. Indeed by periodicity in $\mathcal{N}_\alpha$, one has to consider only sequences of monotone functions defined on $[0, \frac{1}{\alpha})$ and bounded by the functions u and v. Helly's selection theorem then states the existence of a pointwise convergent subsequence which, a fortiori, converges in the Hausdorff topology.

It remains to show that every extended circle map G is continuous in the Hausdorff topology. Let $\{u_n\}_{n\in\mathbb{N}}$ and u be functions in $\mathcal{N}_\alpha$ such that $\lim_{n\to\infty} d(u_n, u) = 0$. We know that $\lim_{n\to\infty} u_n(x) = u(x)$ for every x in the set of points where u is continuous. The complement of this set is countable because u is increasing. By the Cantor diagonal process, let $\{n_k\}$ be a subsequence such that $\{u_{n_k}\}$ pointwise converges.

We recall that P_ℓ and P_r are projections on left continuous and right continuous functions respectively. By monotony, we have $P_\ell u \leq \lim_{k\to\infty} u_{n_k} \leq P_r u$. By continuity, it results that $P_\ell G u \leq \lim_{k\to\infty} G u_{n_k} \leq P_r G u$ and thus $\lim_{k\to\infty} d(G u_{n_k}, G u) = 0$. Since the subsequence $\{n_k\}$ is arbitrary, we conclude that $\lim_{n\to\infty} d(G u_n, G u) = 0$. Therefore, any extended circle map is continuous in the Hausdorff topology.

3. Properties of the Rotation Number

3.1. Continuous Dependence

In this section, we show that small changes in the mean spacing α or in the mapping F itself induce small changes in the rotation number. These properties are not only interesting on their own but are also useful when one takes into account accuracy problems in real or numerical experiments: small errors in the modelisation process or in input parameters produce small errors in output measurements.

3.1.1. Changes in Mean Spacing. To prove continuous dependence of the rotation number on the mean spacing, we assume that F is defined and has its four properties in the set

$$\mathcal{M}^c_{\alpha,\alpha'} = \{u \in \mathcal{B} \ : \ \exists c \geq 0 \ : \ \forall y \geq 0, \ \lfloor \alpha y - c \rfloor \leq T^{-y} u - u \leq \lceil \alpha' y + c \rceil\},$$

where $0 \leq \alpha \leq \alpha' \leq L$ and $c \leq L$ for some $L > 0$. (Again, because of properties of extended circle maps, every set $\mathcal{M}^c_{\alpha,\alpha'}$ is invariant under F.) This additional assumption holds for coupled maps of Section 1.2.1, and also holds for FK models of Section 1.2.2 provided that $\varepsilon \leq \varepsilon_L$ as in Lemma 1.2.

Proposition 3.1. *Let $\{\alpha_n\}_{n\in\mathbb{N}}$ be a sequence in $(0, L]$ converging to $\alpha > 0$, and let ν_n and ν_α be the rotation numbers of F in $\mathcal{M}_{\alpha_n}$ and $\mathcal{M}_\alpha$ respectively. Then $\lim\limits_{n\to\infty} \nu_n = \nu_\alpha$.*

Note that the continuity at $\alpha = 0$ cannot be shown by our method, which uses travelling waves. However, we believe that the rotation number is continuous at $\alpha = 0$.

Proof. As a preliminary result, we show that the previous additional assumption implies that the sequence $\{\nu_n\}_{n\in\mathbb{N}}$ is bounded. Let $\alpha_- = \inf_{n\in\mathbb{N}} \alpha_n, \alpha_+ = \sup_{n\in\mathbb{N}} \alpha_n$, and let the functions defined pointwise by $\varphi^- = \min\{\varphi^-_{\alpha_-}, \varphi^-_{\alpha_+}\}$ and $\varphi^+ = \max\{\varphi^+_{\alpha_-}, \varphi^+_{\alpha_+}\}$. For every $n \in \mathbb{N}$, we have $\varphi^- \leq \varphi^-_{\alpha_n} < \varphi^+_{\alpha_n} \leq \varphi^+$. Applying F, then J, and using relation (5), we deduce that $J(F\varphi^-) \geq -\frac{\nu_n}{\alpha_n} \geq -\frac{\nu_n}{\alpha_n} - \frac{1}{\alpha_n} \geq J(F\varphi^+)$. But $J(F\varphi^-)$ and $J(F\varphi^+)$ both are finite because $\varphi^-, \varphi^+ \in \mathcal{M}^0_{\alpha_-,\alpha_+}$ and the set $\mathcal{M}^0_{\alpha_-,\alpha_+}$ is invariant by F. Consequently, the sequence $\{\nu_n\}_{n\in\mathbb{N}}$ is bounded.

By compactness, we consider a convergent subsequence $\{\nu_{n_k}\}_{k\in\mathbb{N}}$ together with ν_∞, its limit. By Theorem 2.2, let $\phi_{n_k} \in \mathcal{N}_{\alpha_{n_k}}$ be such that $F\phi_{n_k} = T^{-\frac{\nu_{n_k}}{\alpha_{n_k}}} \phi_{n_k}$. By homogeneity, we can assume that $J(\phi_{n_k}) = 0$. Thus $\varphi^- \leq \phi_{n_k} \leq \varphi^+$ for all $k \in \mathbb{N}$.

Applying Lemma A.2 of Appendix A, we deduce the existence of a subsequence, also denoted $\{n_k\}_{k\in\mathbb{N}}$, and a function $\phi_\infty \in \mathcal{N}_\alpha$ such that $\lim_{k\to\infty} \phi_{n_k} = \phi_\infty$. By continuity of F and Lemma A.1, we obtain $F\phi_\infty(x) = T^{-\frac{\nu_\infty}{\alpha}}\phi_\infty(x)$ for every x where $T^{-\frac{\nu_\infty}{\alpha}}\phi_\infty$ is continuous. By density, we can apply P_r to obtain, using continuity, the relation $FP_r\phi_\infty = T^{-\frac{\nu_\infty}{\alpha}} P_r\phi_\infty$ By uniqueness of the rotation number in $\mathcal{M}_\alpha$, it follows that $\nu_\infty = \nu_\alpha$. In particular, ν_∞ does not depend on the subsequence $\{n_k\}$ provided that $\{\nu_{n_k}\}$ is convergent. We conclude that $\lim_{n\to\infty} \nu_n = \nu_\alpha$. □

In addition to convergence of the rotation numbers $\nu_n \to \nu_\alpha$ when $\alpha_n \to \alpha$, this proof shows that a subsequence of solutions ϕ_{n_k}, with $J(\phi_{n_k}) = 0$, of the travelling wave equation in $\mathcal{N}_{\alpha_{n_k}}$ can be extracted which converges, in the Hausdorff topology, to a solution of the travelling wave equation in $\mathcal{N}_\alpha$.

3.1.2. Changes in the Mapping. Motivated by changes in the distribution function of coupled maps, changes in the mapping (with α kept fixed) should be measured in the Hausdorff topology; see Section 2.3.2. In this topology, by assuming uniform convergence of a sequence of extended circle maps, one can prove the convergence of the corresponding rotation numbers.

Proposition 3.2. *Assume that* $\lim_{n\to\infty} \sup_{u\in\mathcal{N}_\alpha} d(F_n u, Fu) = 0$ *where* F_n *(*$n \in \mathbb{N}$*) and* F *are extended circle maps defined in* $\mathcal{N}_\alpha$ *for some* $\alpha > 0$*. Let* ν_n *(*$n \in \mathbb{N}$*) and* ν_α *be the rotation numbers associated with* F_n*(*$n \in \mathbb{N}$*) and* F*. Then* $\lim_{n\to\infty} \nu_n = \nu_\alpha$.

Proof. Similar to the previous proof, we start by showing that the sequence $\{\nu_n\}_{n\in\mathbb{N}}$ is bounded. Let $\varepsilon_n = \sup_{u\in\mathcal{N}_\alpha} d(F_n u, Fu)$. By definition of $d(\cdot,\cdot)$ and using that $\varphi^-_\alpha \in \mathcal{N}_\alpha$, we have $T^{\frac{\varepsilon}{\alpha}} F\varphi^-_\alpha \leq F_n\varphi^-_\alpha \leq T^{-\frac{\varepsilon}{\alpha}} F\varphi^-_\alpha$ provided that n is large enough. By applying J and by using relation (5), we conclude that $\{\nu_n\}_{n\in\mathbb{N}}$ is bounded.

We now consider a convergent subsequence $\{\nu_{n_k}\}_{k\in\mathbb{N}}$ and its limit ν_∞. By periodicity in $\mathcal{N}_\alpha$, Helly's selection theorem can be applied to the corresponding solutions of the travelling wave equations $\{\phi_{n_k}\}_{k\in\mathbb{N}}$ to obtain a subsequence, also denoted $\{\phi_{n_k}\}_{k\in\mathbb{N}}$, such that $\phi_{n_k}\in\mathcal{N}_\alpha$,

$$F_{n_k}\phi_{n_k} = T^{-\frac{\nu_{n_k}}{\alpha}}\phi_{n_k}, \quad \forall k\in\mathbb{N}, \tag{6}$$

and $\lim\limits_{k\to\infty}\phi_{n_k} = \phi_\infty \in \mathcal{N}_\alpha$.

Let $\phi = P_r\phi_\infty$. We have $d(F_{n_k}\phi_{n_k}, F\phi) \le \varepsilon_n + d(F\phi_{n_k}, F\phi)$ and $\lim\limits_{k\to\infty} d(F\phi_{n_k}, F\phi) = 0$ by continuity of F. Therefore $\lim\limits_{k\to\infty} F_{n_k}\phi_{n_k}(x) = F\phi(x)$ if $F\phi$ is continuous at x. On the other hand, by Lemma A.1, we have $\lim\limits_{k\to\infty} T^{-\frac{\nu_{n_k}}{\alpha}}\phi_{n_k}(x) = T^{-\frac{\nu_\infty}{\alpha}}\phi(x)$ at all continuity points of $T^{-\frac{\nu_\infty}{\alpha}}\phi$. By density, one can take the limit $k\to\infty$ in (6) and then apply P_r to obtain $F\phi = T^{-\frac{\nu_\infty}{\alpha}}\phi$, from which the desired conclusion follows by uniqueness of the rotation number. □

In order to justify the assumption of convergence in the Hausdorff topology, we obtain conditions on the examples' parameters which ensure that $\lim\limits_{n\to\infty}\sup\limits_{u\in\mathcal{N}_\alpha} d(F_n u, Fu) = 0$ but not necessarily a stronger convergence. In particular, this happens to be the case for the coupled maps of Section 1.2.1. (The symbol H stands for the Heaviside function.)

Lemma 3.3. *Assume that* $\lim\limits_{n\to\infty} f_n = f$ *where* f_n *(*$n\in\mathbb{N}$*) and* f *are lifts of circle maps, and let* h_n *(*$n\in\mathbb{N}$*) and* h *be distribution functions such that*

(i) there exists a function $g\in\mathcal{B}$ *such that* $|H-h_n|\le g$ *(*$n\in\mathbb{N}$*) and such that the integral* $\int_{\mathbb{R}} g(x)dx$ *exists,*
(ii) $\lim\limits_{n\to\infty} d(h_n, h) = 0$.

Then the mappings $F_n u := h_n * f_n\circ u$ *and* $Fu := h * f\circ u$ *are both defined on* $\mathcal{M}_\alpha$ *(*$\alpha>0$*), and we have* $\lim\limits_{n\to\infty}\sup\limits_{u\in\mathcal{N}_\alpha} d(F_n u, Fu) = 0$.

Assumption (ii) allows us to consider sequences of discrete distribution functions converging to a continuous distribution function and vice versa. Proposition 3.2 and Lemma 3.3 imply that the rotation number of a coupled map with a continuous distribution function can be approximated, with arbitrary accuracy, by the rotation number of a coupled map with a discrete distribution function, and vice versa.

Coupled maps for which the distribution function is the Heaviside function ($h=H$) write $Fu = f\circ u$. In this case, we have $\nu_\alpha = \nu_f$ for every α where $\nu_f = \lim_{t\to\infty}\frac{f^t(x)}{t}$ is the rotation number of the lift of a circle map f. Lemma 3.3 (when applied with $h_n = H$) then implies the continuous dependence of the rotation number of lifts of circle maps in the pointwise topology. However, for a sequence of lifts of circle maps, uniform convergence is equivalent to pointwise convergence. Therefore, a special consequence of Lemma 3.3 and Proposition 3.2 is the continuous dependence of the rotation number of circle maps in the C^0-topology, e.g., Proposition 11.1.6 in [14].

Proof of the Lemma. The first part of the proof of Proposition B.1 in Appendix B shows that both integrals $\int_{\mathbb{R}} x dh_n(x)$ and $\int_{\mathbb{R}} x dh(x)$ exist. Thus, both mappings F_n ($n \in \mathbb{N}$) and F are well-defined. We have

$$d(h_n * f_n \circ u, h * f \circ u) \leq d(h_n * f_n \circ u, h * f_n \circ u) + d(h * f_n \circ u, h * f \circ u). \quad (7)$$

Since f_n and f are continuous and increasing functions, the convergence $f_n \to f$ is uniform on every bounded interval, and thus on the whole line by periodicity (see proof of the next lemma for a detailed similar argument in FK models). Let us define $\delta_n = \sup_{x\in\mathbb{R}} |f_n(x) - f(x)|$. We have $\sup_{x\in\mathbb{R}} |h * f_n(x) - h * f(x)| \leq \delta_n$ for every $u \in \mathcal{N}_\alpha$ and the second term in (7) is under control.

In order to control the first term, it suffices to prove that $\lim_{n\to\infty} \sup_{u\in\mathcal{N}_\alpha} d(h_n * u, h * u) = 0$. By contradiction, assume the existence of $\varepsilon > 0$ and of a subsequence $\{u_{n_i}\}_{i\in\mathbb{N}}$ of functions in $\mathcal{N}_\alpha$ such that

$$d(h_{n_i} * u_{n_i}, h * u_{n_i}) > \varepsilon, \quad \forall i \in \mathbb{N}. \quad (8)$$

Since we have $d(h_{n_i} * u_{n_i}, h * u_{n_i}) = d(h_{n_i} * T^y u_{n_i}, h * T^y u_{n_i})$ for any $y \in \mathbb{R}$, we can assume that the sequence $\{u_{n_i}(0)\}$ is bounded. By Lemma A.2, there exists a pointwise convergent subsequence of $\{u_{n_i}\}$ which we also denote by $\{u_{n_i}\}$. Let u be the limit. We have $\lim_{i\to\infty} d(u_{n_i}, u) = 0$.

For every $i \in \mathbb{N}$, we have

$$d(h_{n_i} * u_{n_i}, h * u_{n_i}) \leq d(h_{n_i} * u_{n_i}, h_{n_i} * u) + d(h_{n_i} * u, h * u) + d(h * u, h * u_{n_i}).$$

Moreover, for every distribution function h and every pair of functions $u, n \in \mathcal{N}_\alpha$, we have

$$d(h * u, h * v) \leq d(u, v).$$

Indeed, if $d(u, v) < \delta$, then we have $T^\delta v - \delta \leq u \leq T^{-\delta} v + \delta$. By convoluting by h, it follows that $T^\delta h * v - \delta \leq h * u \leq T^{-\delta} h * v + \delta$. Hence, we have $d(h * u, h * v) \leq \delta$. Since δ is arbitrary, we conclude that $d(h * u, h * v) \leq d(u, v)$.

Consequently, for every $i \in \mathbb{N}$, we have

$$d(h_{n_i} * u_{n_i}, h * u_{n_i}) \leq d(u_{n_i}, u) + d(h_{n_i} * u, h * u) + d(u, u_{n_i}),$$

and the property $\lim_{i\to\infty} d(u_{n_i}, u) = 0$ and Proposition B.1 imply that $\lim_{i\to\infty} d(h_{n_i} * u_{n_i}, h * u_{n_i}) = 0$, which contradicts relation (8). □

Uniform convergence in the Hausdorff topology can also be shown in FK models. Actually, since their dynamics only involves a finite number of sites, i.e., $Fu(x)$ depends only on $u(x-1)$, on $u(x)$, and on $u(x+1)$, one can show that the convergence holds in the uniform topology.

Lemma 3.4. *Assume pointwise convergence of partial derivatives, i.e.,* $\lim_{n\to\infty} (g_n)'_1 = g'_1$ *and* $\lim_{n\to\infty} (g_n)'_2 = g'_2$ *where* g_n *(*$n \in \mathbb{N}$*) and* g *are generating functions. Assume also that*

$\lim_{n\to\infty} \varepsilon_n = \varepsilon$ and $\lim_{n\to\infty} D_n = D$ where ε_n ($n \in \mathbb{N}$) is such that

$$F_n u(x) = u(x) - \varepsilon_n((g_n)'_2(u(x-1), u(x)) + (g_n)'_1(u(x), u(x+1)) + D_n)$$

is increasing on $\mathcal{N}_\alpha$ and ε is such that F is increasing on the same set. Then, we have

$$\lim_{n\to\infty} \sup_{u\in\mathcal{N}_\alpha, x\in\mathbb{R}} |F_n u(x) - F u(x)| = 0.$$

Proof. As in the proof of Lemma 1.2, let G_n ($n \in \mathbb{N}$) be the maps such that $F_n u(x) = G_n(u(x-1), u(x), u(x+1))$ and define the map G similarly. By assumption, we have pointwise convergence $\lim_{n\to\infty} G_n = G$. Since $\mathcal{N}_\alpha = \mathcal{M}^0_\alpha$, we have to prove that this convergence is indeed uniform on

$$\mathcal{D}_\alpha = \{(x, y, z) \in \mathbb{R}^3 \ :\ y \in [0, 1], \lfloor\alpha\rfloor \le y - x, z - y \le \lceil\alpha\rceil\}.$$

By contradiction, assume the existence of $\eta > 0$ and of a subsequence $\{n_i\}_{i\in\mathbb{N}}$ such that

$$\sup_{(x,y,z)\in\mathcal{D}_\alpha} |G_{n_i}(x, y, z) - G(x, y, z)| \ge \eta.$$

By continuity and compactness, for every $i \in \mathbb{N}$, there exists $(x_i, y_i, z_i) \in \mathcal{D}_\alpha$ such that $|G_{n_i}(x_i, y_i, z_i) - G(x_i, y_i, z_i)| \ge \eta$. Endow $\mathbb{R}^3$ with uniform norm and let $\delta > 0$ be such that $(x, y, z), (\bar{x}, \bar{y}, \bar{z}) \in \mathcal{D}_\alpha$ with $\|(x, y, z) - (\bar{x}, \bar{y}, \bar{z})\| < \delta$ implies $|G(x, y, z) - G(\bar{x}, \bar{y}, \bar{z})| < \eta$.

By compactness and by taking a subsequence if necessary, the sequence $\{(x_i, y_i, z_i)\}_{i\in\mathbb{N}}$ is convergent. By again taking a subsequence if necessary, we can assume that either $G_{n_i}(x_i, y_i, z_i) - G(x_i, y_i, z_i) \ge \eta$ for all $i \in \mathbb{N}$ or $G_{n_i}(x_i, y_i, z_i) - G(x_i, y_i, z_i) \le -\eta$ for all $i \in \mathbb{N}$.

Suppose the first case is true. The second case can be solved similarly. Let $(x_\infty, y_\infty, z_\infty) = \lim_{i\to\infty}(x_i, y_i, z_i)$. Given $m \in \mathbb{N}$, define the following numbers:

$$\bar{x}_m = \sup_{i>m} x_i, \qquad \bar{y}_m = \sup_{i>m} y_i, \qquad \text{and} \qquad \bar{z}_m = \sup_{i>m} z_i.$$

Let m be sufficiently large so that $\|(\bar{x}_m, \bar{y}_m, \bar{z}_m) - (x_\infty, y_\infty, z_\infty)\| < \delta$. For every $i > m$, we have $(x_i, y_i, z_i) \le (\bar{x}_m, \bar{y}_m, \bar{z}_m)$, and hence

$$G_{n_i}(\bar{x}_m, \bar{y}_m, \bar{z}_m) - G(x_i, y_i, z_i) \ge G_{n_i}(x_i, y_i, z_i) - G(x_i, y_i, z_i) \ge \eta,$$

which implies the inequality $G(\bar{x}_m, \bar{y}_m, \bar{z}_m) - G(x_\infty, y_\infty, z_\infty) \ge \eta$. We have a contradiction with uniform continuity. □

3.2. Extended Circle Maps with Vanishing Rotation Number

In this section, a symmetry condition on extended circle maps is given which implies that the rotation number is zero. We show that this condition holds for our examples provided that suitable assmuptions on their parameters are satisfied. In particular, in FK models, the only assumption is $D = 0$. In other words, one obtains the statement of the

Aubry-Mather Theorem [12] on the existence of fixed points for every mean spacing $\alpha \in \mathbb{R}$.

Theorem 3.5. *If there exists a lift of a circle map f such that the following relation holds,*

$$\int_0^{\frac{1}{\alpha}} (Fu - u)d(f \circ u) = 0, \tag{9}$$

for every continuous function $u \in \mathcal{N}_\alpha$ ($\alpha > 0$), then the rotation number $\nu_\alpha = 0$.

Notice that a similar condition has been obtained in [10] that forces the front velocity to vanish.

We use the symbol $\int_a^b u d\mu$ instead of $\int_A u d\mu$ when the integral does not depend on the fact that A is the closed, semi-open, or open interval between a and b.

Proof. Let $\phi \in P_r(\mathcal{N}_\alpha)$ be a solution of the travelling wave equation. Given $n \in \mathbb{N}$, let $\phi_n \in \mathcal{N}_\alpha$ be defined by $\phi_n(x) = n\int_{[0,\frac{1}{n}]}\phi(x+y)dy$ for all $x \in \mathbb{R}$. This function is continuous and satisfies the inequalities $\phi \leq \phi_n \leq T^{-\frac{1}{n}}\phi$. By right continuity of ϕ, it follows that $\lim\limits_{n\to\infty} \phi_n = \phi$.

Consider the quantity $\Delta := \int_{\phi(0)}^{\phi(\frac{1}{\alpha})} x df(x)$ where f is given in (9). Since every ϕ_n is increasing and continuous, we can apply a change of variable for Lebesgue-Stieltjes integrals [17] to obtain

$$\Delta = \lim_{n\to\infty} \int_{\phi_n(0)}^{\phi_n(\frac{1}{\alpha})} x df(x) = \lim_{n\to\infty} \int_0^{\frac{1}{\alpha}} \phi_n d(f \circ \phi_n).$$

By relation (9), by monotony, and by definition of ϕ, we have

$$\Delta = \lim_{n\to\infty} \int_0^{\frac{1}{\alpha}} F\phi_n d(f \circ \phi_n) \geq \lim_{n\to\infty} \int_0^{\frac{1}{\alpha}} F\phi d(f \circ \phi_n) = \lim_{n\to\infty} \int_0^{\frac{1}{\alpha}} T^{-\frac{\nu_\alpha}{\alpha}}\phi d(f \circ \phi_n).$$

By applying Lemma A.3 to the latter, we obtain

$$\Delta \geq \int_{(0,\frac{1}{\alpha}]} P_\ell T^{-\frac{\nu_\alpha}{\alpha}}\phi d(f \circ \phi) = P_\ell B(\frac{\nu_\alpha}{\alpha}),$$

where $B(\nu) := \int\limits_{(0,\frac{1}{\alpha}]} T^{-\nu}\phi d(f \circ \phi)$.

Similarly, fix $\varepsilon > 0$. For every $n > \frac{1}{\varepsilon}$, we have $\phi_n \leq T^{-\frac{1}{n}}\phi \leq T^{-\varepsilon}\phi$, and then

$$\Delta \leq \lim_{n\to\infty} \int_0^{\frac{1}{\alpha}} T^{-\varepsilon} F\phi d(f \circ \phi_n) = \int_{(0,\frac{1}{\alpha}]} T^{-\frac{\nu_\alpha}{\alpha}} T^{-\varepsilon} P_\ell \phi d(f \circ \phi).$$

Consequently, we have

$$\Delta \leq \lim_{\varepsilon \to 0} \int_{(0,\frac{1}{\alpha}]} T^{-\frac{\nu_\alpha}{\alpha}} T^{-\varepsilon} P_\ell \phi d(f \circ \phi) = \int_{(0,\frac{1}{\alpha}]} T^{-\frac{\nu_\alpha}{\alpha}} \phi d(f \circ \phi) = B\left(\frac{\nu_\alpha}{\alpha}\right),$$

because $P_r P_\ell u = P_r u$ for every increasing function u and $P_r \phi = \phi$. Therefore, we have proved that

$$P_\ell B\left(\frac{\nu_\alpha}{\alpha}\right) \leq \Delta \leq B\left(\frac{\nu_\alpha}{\alpha}\right). \tag{10}$$

The theorem then results from the next statement.

Lemma 3.6. *For every $\nu > 0$, we have $B(-\nu) < \Delta < B(\nu)$.*

Since the function ϕ is increasing, so is the function $\nu \mapsto B(\nu)$. Lemma 3.6 thus implies that $\nu_\alpha = 0$, and the Theorem follows. Indeed, by using contradiction, if $\nu_\alpha > 0$, then the Lemma would imply that $\Delta < P_\ell B(\nu_\alpha)$, which is impossible by relation (10). So $\nu_\alpha \leq 0$. Similarly, if $\nu_\alpha < 0$, we would have $B(\nu_\alpha) < \Delta$ which also contradicts relation (10).

Proof of the Lemma. Given an increasing function u, we define the set of increase points of u in $(0, \frac{1}{\alpha}]$ as follows:

$$E(u) = \left\{ x \in \left(0, \frac{1}{\alpha}\right] \ : \ u(x-\delta) < u(x+\delta) \ \forall \delta > 0 \right\}.$$

The monotony of f implies that $E(f \circ \phi) \subset E(\phi)$. Moreover $f \circ \phi(\frac{1}{\alpha}) = f \circ \phi(0) + 1$ implies that $E(f \circ \phi) \neq \emptyset$. We consider separately the cases where this set is finite and where it is infinite.

First, assume that $E(f \circ \phi) = \{x_i\}_{i=1}^n$ is a finite set and let us prove that $\Delta < B(0)$. Using right continuity of ϕ and letting $x_0 = 0$, we have $f \circ P_\ell \phi(x_i) = f \circ \phi(x_{i-1})$ for every $i \in \{1, \ldots, n\}$ and $f \circ \phi(x_n) = f \circ \phi(\frac{1}{\alpha})$. It results that

$$B(0) = \sum_{i=1}^{n} \phi(x_i)(f \circ \phi(x_i) - f \circ \phi(x_{i-1})),$$

and then

$$B(0) - \Delta = \sum_{i=1}^{n} \left(\phi(x_i)(f \circ \phi(x_i) - f \circ \phi(x_{i-1})) - \int_{\phi(x_{i-1})}^{\phi(x_i)} x df(x) \right).$$

The monotony of f and ϕ imply that $B(0) - \Delta \geq 0$. Moreover, since $f \circ \phi(x_i) > f \circ \phi(x_{i-1})$, we deduce that $B(0) - \Delta > 0$.

By taking also into account the monotony of $B(\nu)$ with ν, we have proved that $\Delta < B(0) \leq B(\nu)$ for every $\nu > 0$. By using similar arguments, one can prove that $B(-\nu) \leq P_\ell B(0) < \Delta$ for every $\nu > 0$.

Assume now that the set of increase points $E(f \circ \phi)$ is infinite. Let $\nu > 0$ be fixed. Since the set $E(f \circ \phi)$ is infinite and included in $(0, \frac{1}{\alpha}]$, there exist $a, b \in (0, \frac{1}{\alpha}) \cap E(f \circ \phi)$ such that $a < b < a + \nu$.

Let $\delta > 0$ be sufficiently small such that $0 < a - \delta < a + \delta \le \frac{1}{\alpha}$, $b + \delta < a + \nu - \delta$, and $a + \delta < b - \delta$. By using monotony, we have

$$\begin{aligned} B(\nu) - B(0) &= \int_{(0,\frac{1}{\alpha}]} (T^{-\nu}\phi - \phi) d(f \circ \phi) \\ &\ge \int_{(a-\delta,a+\delta]} (T^{-\nu}\phi - \phi) d(f \circ \phi) \\ &\ge (\phi(a+\nu-\delta) - \phi(a+\delta)) \int_{(a-\delta,a+\delta]} d(f \circ \phi) \\ &\ge (\phi(b+\delta) - \phi(b-\delta))((f \circ \phi)(a+\delta) - (f \circ \phi)(a-\delta)). \end{aligned}$$

Since $b \in E(f \circ \phi)$ implies $b \in E(\phi)$, the last term is positive, and hence $B(\nu) > B(0)$.

Therefore, in order to prove that $\Delta < B(\nu)$, it remains to show that $\Delta \le B(0)$. Since ϕ_n is continuous and $\phi_n \le T^{-\varepsilon}\phi$ when $n > \frac{1}{\varepsilon}$, we have

$$\Delta = \lim_{n\to\infty} \int_0^{\frac{1}{\alpha}} \phi_n d(f \circ \phi_n) \le \lim_{n\to\infty} \int_0^{\frac{1}{\alpha}} T^{-\varepsilon}\phi d(f \circ \phi_n).$$

By applying Lemma A.3 and by taking the limit $\varepsilon \to 0$ on the right-hand side, we obtain

$$\Delta \le \lim_{\varepsilon\to 0} \int_{(0,\frac{1}{\alpha}]} T^{-\varepsilon} P_\ell h(n) \phi d(f \circ \phi) = \int_{(0,\frac{1}{\alpha}]} \phi d(f \circ \phi) = B(0),$$

which is the desired result. The inequality $B(-\nu) < \Delta$ can be proved analogously. □

3.2.1. Application to Coupled Maps. In coupled maps $Fu = h * f \circ u$, condition (9) holds provided that the local map is area-symmetric and the coupling induced by h is symmetric:

Proposition 3.7. *If f is such that $\int_0^1 (f(x) - x)dx = 0$ and if h satisfies $P_\ell h(x) = 1 - h(-x)$ for all $x \in \mathbb{R}$, then, for every $\alpha \in \mathbb{R}$, the rotation number of the corresponding coupled map $Fu = h * f \circ u$ is zero.*

Proof. We show that the coupled map satisfies condition (9) for every $\alpha > 0$. Precisely, we show that $\int_0^{\frac{1}{\alpha}} (h * f \circ u - u) d(f \circ u) = 0$ for every continuous function $u \in \mathcal{N}_\alpha$ and every $\alpha > 0$.

As a preliminary step, we claim that $\int_0^1 (f(x) - x) df(x) = 0$. Indeed, by using integration by parts for Lebesgue-Stieltjes integrals [17] and a change of variable, we have

$$\begin{aligned} \int_0^1 (f(x) - x) df(x) &= \int_0^1 f(x) df(x) + \int_0^1 f(x) dx - [x f(x)]_0^1 \\ &= \int_{f(0)}^{f(1)} x dx + \int_0^1 f(x) dx - f(1) \\ &= \int_0^1 f(x) dx - \frac{1}{2} = \int_0^1 (f(x) - x) dx = 0. \end{aligned}$$

Consequently, for every continuous function $u \in \mathcal{N}_\alpha$, we have $\int_0^{\frac{1}{\alpha}} ud(f \circ u) = \int_0^{\frac{1}{\alpha}} (f \circ u)d(f \circ u)$, and then

$$\int_0^{\frac{1}{\alpha}} (h * f \circ u - u)d(f \circ u) = \int_{\mathbb{R}} \int_0^{\frac{1}{\alpha}} (T^y f \circ u - f \circ u)d(f \circ u)dh(y),$$

where Fubini's theorem has been applied in order to change the order of integration.

Now, the integral over $\mathbb{R}$ splits into the sum of an integral over $\mathbb{R}^-$, an integral over $\{0\}$, and an integral over $\mathbb{R}^+$. The integral over $\{0\}$ vanishes. By a change of variable, by using the assumption $P_\ell h(y) = 1 - h(-y)$ and $T^{-\frac{1}{\alpha}} u = u + 1$, the integral over $\mathbb{R}^+$ becomes

$$\begin{aligned}
&\int_{\mathbb{R}^+} \int_0^{\frac{1}{\alpha}} (T^y f \circ u - f \circ u)d(f \circ u)dh(y) \\
&\quad = \int_{\mathbb{R}^-} \int_y^{\frac{1}{\alpha}+y} (f \circ u - T^y f \circ u)d(T^y f \circ u)d(1 - h(-y)) \\
&\quad = \int_{\mathbb{R}^-} \int_0^{\frac{1}{\alpha}} (f \circ u - T^y f \circ u)d(T^y f \circ u)dh(y).
\end{aligned}$$

Therefore

$$\begin{aligned}
\int_0^{\frac{1}{\alpha}} (h * f \circ u - u)d(f \circ u) &= \int_{\mathbb{R}^-} \int_0^{\frac{1}{\alpha}} (f \circ u - T^y f \circ u)d(T^y f \circ u - f \circ u)dh(y) \\
&= -\frac{1}{2} \int_{\mathbb{R}^-} \int_0^{\frac{1}{\alpha}} d(f \circ u - T^y f \circ u)^2 dh(y) = 0,
\end{aligned}$$

where the last equality follows from the fact that $T^{-\frac{1}{\alpha}} u = u + 1$. □

3.2.2. Application to FK Models. The symmetry condition (9) holds in any FK model without constant force ($D = 0$):

Proposition 3.8. *For every generating function g, every $\alpha > 0$, and every $\varepsilon > 0$, the map*

$$F_\varepsilon u(x) = u(x) - \varepsilon(g_2'(u(x-1), u(x)) + g_1'(u(x), u(x+1))), \quad \forall x \in \mathbb{R},$$

satisfies the condition (9) with $f(x) = x$.

The proof is straightforward. Indeed by using $T^{-\frac{1}{\alpha}} u = u + 1$, we have

$$\begin{aligned}
\int_0^{\frac{1}{\alpha}} (F_\varepsilon u - u)du &= -\varepsilon \int_0^{\frac{1}{\alpha}} g_2'(u(x), u(x+1))du(x+1) \\
&\quad - \varepsilon \int_0^{\frac{1}{\alpha}} g_1'(u(x), u(x+1))du(x)
\end{aligned}$$

$$= -\varepsilon \int_0^{\frac{1}{\alpha}} dg(u(x), u(x+1)) = 0,$$

for every continuous function $u \in \mathcal{N}_\alpha$.

By Lemma 1.2, for every $\alpha > 0$, there exists $\varepsilon_\alpha > 0$ such that the map F_{ε_α} with $D = 0$ is an extended circle map (on $\mathcal{N}_\alpha$). By Proposition 3.8 and Theorem 2.2, we conclude that this map has fixed point in $\mathcal{N}_\alpha$ for every $\alpha \in \mathbb{R}$ (the case $\alpha = 0$ can be achieved easily).

Fixed points of F_{ε_α} with $D = 0$ are stationary points of the functional $\sum_{x\in\mathbb{Z}} g(u(x), u(x+1))$, that is to say, functions such that

$$g_2'(u(x-1), u(x)) + g_1'(u(x), u(x+1)) = 0, \quad \forall x \in \mathbb{Z}.$$

Consequently, we have shown that every generating function has stationary functions in $\mathcal{N}_\alpha$, and a fortiori ordered stationary lattice configuration $\{\phi(x+\sigma)\}_{x\in\mathbb{Z}}$ of bounded width and mean spacing α, for every $\alpha \in \mathbb{R}$. This is nothing else but the statement of the Aubry-Mather Theorem.

4. Concluding Remarks

This paper presents an analysis of the dynamics of extended circle maps. In addition to monotony and continuity, we have assumed two fundamental properties of the mappings. These properties are: commutation with every real horizontal translation and commutation with every integer vertical translation.

This analysis can be adapted in order to apply to mappings which commute with every integer horizontal translation and with every real vertical translation. In other words, one can consider F which are monotone, continuous and such that Fu is increasing when u is increasing, $FT^1u = T^1Fu$ and $FR_\nu u = R_\nu Fu$ for every $\nu \in \mathbb{R}$ and every $u \in \mathcal{M}_\alpha$. Here $(R_\nu u)(x) = u(x) + \nu$ $(x \in \mathbb{R})$ is the vertical translation acting on functions in $\mathcal{B}$.

The mappings defined by $Fu = h * (u \circ f)$, where h is a distribution function and f is a lift of a circle map, are examples of such mappings. In particular, when $h = H$, these maps reduce to $Fu = u \circ f$.

By following the proofs of Sections 2 and 3 and by exchanging the roles of T^ν and R_ν, one can obtain the following results for such mappings:

- The existence and uniqueness of the rotation number $\lim_{t\to\infty} \frac{F^t u(x)}{t} = \nu_\alpha$ in $\mathcal{M}_\alpha$.
- For every $\alpha \neq 0$, the existence of functions $\phi \in \mathcal{O}_\alpha$ such that $F\phi = R_{\nu_\alpha}\phi$. The set $\mathcal{O}_\alpha$ is the subset of $\mathcal{M}_\alpha$ of increasing functions u such that $T^1u = R_\alpha u$.
- The continuous dependence of ν_α with α and with the mapping.
- The symmetry condition that implies $\nu_\alpha = 0$.

Finally, we mention two open problems in the dynamics of (original) extended circle maps. The first problem concerns the existence of fronts in $\mathcal{M}_\alpha$. As in Section 2.3, suppose that two consecutive solutions of $T^{\frac{\nu_\alpha}{\alpha}}Fu = u$ in $\mathcal{N}_\alpha$, namely ϕ_1 and ϕ_2, are such that $\phi_1 < \phi_2$. A front (of velocity 0) is a solution ϕ of the same equation in $\mathcal{M}_\alpha \setminus \mathcal{N}_\alpha$ such that $\phi_1 < \phi < \phi_2$ and $(\phi - \phi_1)(-\infty) = (\phi - \phi_2)(+\infty) = 0$. One additional difficulty when dealing with $\mathcal{M}_\alpha \setminus \mathcal{N}_\alpha$ is the noncompactness of this set.

Indeed, pointwise limits of functions in this set may belong to $\mathcal{N}_\alpha$. In the case where $\alpha = 0$, the existence of fronts, with not necessarily zero velocity, has been proved in [8] for coupled maps.

The second problem concerns the dynamics of functions with different growth rates at $+\infty$ and $-\infty$—in particular, the functions which are at bounded distance from $x \mapsto \alpha x$ for all $x < 0$ and at bounded distance from $x \mapsto \alpha' x$ for all $x > 0$. One may wonder about the existence and uniqueness of a rotation number and the existence of travelling waves. If $0 \leq \alpha \leq \alpha'$, then by monotony of F, their rotation number (if it exists) must be larger than $\max\{\nu_\alpha, \nu_{\alpha'}\}$. This is because every such function is both larger than $x \mapsto \alpha x + c$ for some $c \in \mathbb{R}$ and larger than $x \mapsto \alpha' x + c'$ for some $c' \in \mathbb{R}$.

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Appendix A. Auxiliary Results

Lemma A.1. [8] *Let $\{a_n\}_{n\in\mathbb{N}}$ be a real sequence converging to a real number a and let $\{u_n\}_{n\in\mathbb{N}}$ be a sequence of increasing functions such that $\lim_{n\to\infty} u_n = u$. If u is continuous at $x - a$, then the limit $\lim_{n\to\infty} T^{a_n} u_n(x)$ exists and is equal to $T^a u(x)$.*

The functions in this statement can be defined on bounded or unbounded domains. In [8], the proof was accomplished assuming that $\{u_n\}_{n\in\mathbb{N}}$ is equibounded. However, this restriction is not necessary.

Lemma A.2. *Let $\{\alpha_n\}_{n\in\mathbb{N}}$ be a sequence of real numbers converging to $\alpha > 0$. Then, any sequence $\{u_n\}_{n\in\mathbb{N}}$ of functions $u_n \in \mathcal{N}_{\alpha_n}$ such that the sequence $\{u_n(0)\}_{n\in\mathbb{N}}$ is bounded has a convergent subsequence $\lim_{k\to\infty} u_{n_k} = u \in \mathcal{N}_\alpha$.*

Proof. Given $\delta > 0$, by assumption on $\{u_n\}_{n\in\mathbb{N}}$, we have $\sup_{n\in\mathbb{N}, x\in[-\delta, \frac{1}{\alpha}+\delta]} |u_n(x)| < +\infty$. Moreover, these functions are all increasing. By Helly's selection theorem [15], there exists a convergent subsequence $\lim_{i\to\infty} u_{n_i}(x) = w(x)$ for every $x \in [-\delta, \frac{1}{\alpha} + \delta]$.

Let us consider the function v defined by $v(x) = w(x - \frac{\lfloor \alpha x \rfloor}{\alpha}) + \lfloor \alpha x \rfloor$ for every $x \in \mathbb{R}$. For every $x \in \mathbb{R}$, the quantity $x - \frac{\lfloor \alpha x \rfloor}{\alpha_n} \in [-\delta, \frac{1}{\alpha} + \delta]$ if n is sufficiently large. Therefore, by Lemma A.1, we have

$$\lim_{i\to\infty} u_{n_i}(x) = \lim_{i\to\infty} u_{n_i}\left(x - \frac{\lfloor \alpha x \rfloor}{\alpha_n}\right) + \lfloor \alpha x \rfloor = w\left(x - \frac{\lfloor \alpha x \rfloor}{\alpha}\right) + \lfloor \alpha x \rfloor = v(x),$$

if v is continuous at $x - \frac{\lfloor \alpha x \rfloor}{\alpha}$. In other terms, u_{n_i} converges to v at all points of the line excepted a countable set. By applying the diagonal process, we conclude the existence of a subsequence $\{n_{i_k}\}$ such that $\lim_{k\to\infty} u_{n_{i_k}} = u$ where $u \in \mathcal{N}_\alpha$ and $P_\ell v \le u \le P_r v$. $\square$

Lemma A.3. *Let $a < b \in \mathbb{R}$, ϕ_n ($n \in \mathbb{N}$) be continuous functions on $[a, b]$ such that $\lim_{n\to\infty} \phi_n = \phi$ is of bounded variation. Let also ψ be of bounded variation on $[a, b]$. If ϕ is right continuous, then*

$$\lim_{n\to\infty} \int_{(a,b)} \psi \, d\phi_n = \int_{(a,b]} P_\ell \psi \, d\phi.$$

Proof. By continuity of ϕ_n, we have $\int_{(a,b)} \psi \, d\phi_n = \int_{(a,b)} P_\ell \psi \, d\phi_n$. By using integration by parts and the definition of the Lebesgue-Stieltjes integral, we obtain

$$\int_{(a,b)} P_\ell \psi \, d\phi_n = P_\ell \psi(b)\phi_n(b) - P_r \psi(a)\phi_n(a) - \int_{(a,b)} \phi_n \, d\psi.$$

By Lebesgue convergence theorem, the integral in the right-hand side converges to $\int_{(a,b)} \phi \, d\psi$. By integration by parts and using that ϕ is right continuous, we have

$$\int_{(a,b)} \phi \, d\psi = P_\ell \psi(b) P_\ell \phi(b) - P_r \psi(a)\phi(a) - \int_{(a,b)} P_\ell \psi \, d\phi.$$

Therefore

$$\lim_{n\to\infty} \int_{(a,b)} P_\ell \psi \, d\phi_n = P_\ell \psi(b)(\phi(b) - P_\ell \phi(b)) + \int_{(a,b)} P_\ell \psi \, d\phi = \int_{(a,b]} P_\ell \psi \, d\phi,$$

which is the desired result. $\square$

Appendix B. On Properties of Lebesgue-Stieltjes Integrals

The goal of this section is to prove the following proposition. We recall that the quantity $d(\cdot, \cdot)$ was introduced in Section 2.3.2.

Proposition B.1. *Let h_n ($n \in \mathbb{N}$) and h be distribution functions such that*

(i) there exists a function $g \in \mathcal{B}$ such that $|H - h_n| \le g$ ($n \in \mathbb{N}$) and such that the integral $\int_{\mathbb{R}} g(x)dx$ exists,
(ii) $\lim_{n\to\infty} d(h_n, h) = 0$.

Then, for every $\alpha > 0$ and every $u \in \mathcal{N}_\alpha$, the functions $h_n * u$ ($n \in \mathbb{N}$) and $h * u$ are well defined and we have

$$\lim_{n\to\infty} d(h_n * u, h * u) = 0.$$

To obtain this result, we need an auxiliary statement.

Lemma B.2. *Let h be a distribution function and let H be the Heaviside function. The integral $\int_{\mathbb{R}} x dh(x)$ exists iff $\int_{\mathbb{R}}(H - h)(x)dx$ exists. Moreover, when they exist, these integrals are equal and we have*

$$\lim_{x\to+\infty} x(1 - h(x)) = \lim_{x\to-\infty} xh(x) = 0.$$

Proof. We prove that the existence of the integral $\int_{\mathbb{R}} x dh(x)$ implies the desired limits. By contradiction, assume that $\int_{\mathbb{R}} |x| dh(x)$ exists and $\limsup_{x\to+\infty} x(1 - h(x)) = \varepsilon > 0$. Since in addition $h(x) \to 1$ when $x \to +\infty$, by induction, one constructs an increasing unbounded sequence $\{x_k\}_{k\in\mathbb{N}}$ of positive numbers such that $x_k(1 - h(x_k)) > \frac{\varepsilon}{2}$ and $x_k(1 - h(x_{k+1})) < \frac{\varepsilon}{4}$ for all $k \in \mathbb{N}$. As a consequence, we have

$$\begin{aligned}\int_{\mathbb{R}} |x| dh(x) &\geq \int_{(x_1,+\infty)} |x| dh(x) = \sum_{k=1}^{+\infty} \int_{(x_k,x_{k+1}]} |x| dh(x) \\ &\geq \sum_{k=1}^{+\infty} x_k(h(x_{k+1}) - h(x_k)) = +\infty,\end{aligned}$$

which gives a contradiction. The proof that $\lim_{x\to-\infty} xh(x) = 0$ follows similarly.

Moreover, since $\int_{\mathbb{R}} x dH(x) = 0$, we have $\int_{\mathbb{R}} x dh(x) = \int_{\mathbb{R}} x d(h - H)(x)$ whenever these integrals exist. By assuming the existence of $\int_{\mathbb{R}} x dh(x)$ and by using integration by parts for Lebesgue-Stieltjes integrals, we obtain the existence of $\int_{\mathbb{R}}(H - h)(x)dx$ and

$$\begin{aligned}\int_{\mathbb{R}} (H - h)(x)dx &= \int_{\mathbb{R}} x d(h - H)(x) + \int_{\mathbb{R}} d(x(H - h)(x)) \\ &= \int_{\mathbb{R}} x dh(x) + \lim_{x\to+\infty} x(1 - h(x)) + \lim_{x\to-\infty} xh(x),\end{aligned}$$

which gives the desired equality.

By definition of the Lebesgue-Stieltjes integral, the integral $\int_{\mathbb{R}}(H - h)(x)dx$ exists iff the integral $\int_{\mathbb{R}} |H - h|(x)dx$ exists. It is left to the reader to show that the existence of the latter implies the desired limits. Finally, arguments similar to the previous ones can be used to show that the existence of $\int_{\mathbb{R}}(H - h)(x)dx$ implies that of $\int_{\mathbb{R}} x dh(x)$. □

Proof of Proposition B.1. The assumptions and Lebesgue's dominated convergence theorem imply the existence of the integrals $\int_{\mathbb{R}}(H - h_n)(x)dx$ ($n \in \mathbb{N}$) and $\int_{\mathbb{R}}(H - h)(x)dx$. By Lemma B.2, we deduce the existence of the integrals $\int_{\mathbb{R}} x dh_n(x)$ and $\int_{\mathbb{R}} x dh(x)$. Hence the functions $h_n * u$ and $h * u$ are well defined for every $u \in \mathcal{N}_\alpha$.

As a consequence, the function $(h_n - h) * u = h_n * u - h * u$ is well defined. By using integration by parts and Lemma B.2, we obtain $(h_n - h) * u = u * (h_n - h)$. In order to obtain the result of the Proposition, we then need to show that $\lim_{n\to\infty} u * (h_n - h)(x) = 0$ a.e.

To that purpose, we assume, without loss of generality, that the function g satisfies the following relation:

$$g(x) = \sup_{n\in\mathbb{N}} |(H - h_n)(x)|, \quad \forall x \in \mathbb{R},$$

and we prove the existence of the integral $\int_{\mathbb{R}} g(x)du(x)$ for every $u \in \mathcal{N}_\alpha$. Since u is increasing and g is nonnegative, in order to prove the existence, one only has to obtain a finite upper bound.

The function g is decreasing on $[0, +\infty)$. Thus, we have

$$\int_{[0,+\infty)} g(x)du(x) = \sum_{k=0}^{+\infty} \int_{[\frac{k}{\alpha},\frac{k+1}{\alpha})} g(x)du(x) \le \sum_{k=0}^{+\infty} g\left(\frac{k}{\alpha}\right) = g(0) + \sum_{k=1}^{+\infty} g\left(\frac{k}{\alpha}\right).$$

By using again that g is decreasing on $[0, +\infty)$, we obtain

$$\sum_{k=1}^{+\infty} g\left(\frac{k}{\alpha}\right) \le \alpha \sum_{k=0}^{+\infty} \int_{[\frac{k}{\alpha},\frac{k+1}{\alpha})} g(x)dx = \alpha \int_{[0,+\infty)} g(x)dx,$$

and the existence of $\int_{[0,+\infty)} g(x)dx$ implies the existence of the integral $\int_{[0,+\infty)} g(x) du(x)$. The existence of $\int_{(-\infty,0]} g(x)du(x)$ can be proved similarly.

Finally, assumption *(ii)*, the existence of $\int_{\mathbb{R}} g(x)du(x)$, the inequality $|h_n - h| \le 2g$, and Lebesgue's dominated convergence theorem imply that $\lim_{n\to\infty} \int_{\mathbb{R}} (h_n - h)(x - y)du(y) = 0$ for every $x \in \mathbb{R}$ and every $u \in \mathcal{N}_\alpha$. The proof of the proposition is complete. □

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