The CML2004 Project

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Coupled map lattices (CML) are basic models for the time evolution of nonlinear systems which, above all, are extended in space or involve many individual units. The characteristic features of CML are

- discrete time dynamics
- discrete nature of the underlying space (lattice or network)
- the local variables consist of real numbers or real vectors.

Formally speaking, a CML is a discrete time dynamical system generated by a mapping acting on real (vector) sequences. The typical and most studied example is the model introduced by Kaneko in 1983 and given by the following iterations

$$u_s^{t+1} = (1-\epsilon) f(u_s^t) + \frac{\epsilon}{2}\left(f(u_{s-1}^t) + f(u_{s+1}^t)\right) \quad t \in \mathbb{N},\ \epsilon \in [0,1]$$

where $u_s^t \in \mathbb{R}$ and f is a real mapping.

Depending on the context, the configurations $\{u_s^t\}$ represent the spatial profile of a chemical concentration, of a population density, of a velocity field, etc. In these cases, the configurations are bounded sequences, sometimes finite or periodic. Some systems however require unbounded configurations. This is the case for instance in the Frenkel-Kontorova model of particle chains where u_s^t represents the position along the real line of the sth particle, see the chapters on monotone dynamics [Floría, Baesens and Gómez-Gardenes], [Baesens] and [Coutinho].[1]

As shown by the basic model, the dynamics of a CML is governed by two competing terms; an individual nonlinear reaction represented by f and a spatial interaction (coupling) with variable intensity ϵ. In the basic model, the interaction is a convolution operator which represents a diffusive coupling. These two terms are applied successively, a characteristic feature of CML, and

[1] We refer to the chapters by using the name(s) of their author(s).

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this ensures that bounded initial configurations remain bounded (provided that f is bounded). However, CML are not restricted to such composition nor to convolution couplings and many other models have been considered.

Their simple formulation make CML a paradigm of nonlinear spatially extended dynamical systems. In particular, CML are specially designed to facilitate computer simulations over large space-time domains. The simulations exhibit an extraordinary large panel of behaviours upon changes in the local map and in the interaction (or simply in their parameters). This diverse phenomenology motivated the application of CML to the simulation of real systems. For instance, a spectacular application pointed out to us by Pierre Guiraud is afforded by the simulation of cloud formation by a CML derived from fluid dynamics equations[2].

In the endeavour to describe CML analytically, various methods, techniques and tools have been borrowed from the theory of Dynamical Systems (stability analysis, Lyapunov exponents, bifurcations, symbolic dynamics, etc.). Some results have been confirmed or obtained in a rigorous mathematical framework (e.g. global and partial synchronisation, front dynamics, etc.). As far as Mathematics is concerned, CML form a proper source of problems since they are dynamical systems with infinite dimensional phase space and since they do not satisfy the usual assumptions on dynamics for large physically relevant sets of parameters (e.g. uniform hyperbolicity, prescribed symbolic dynamics, etc.).

For a more complete exposition of the origins of CML, of their motivations, and for an overview of problems, we refer to chapter introductions, especially of [Bunimovich] and [Just and Schmüser].

The purposes of the meeting CML2004 were to present a survey of the theory of CML and of related spatially extended systems (lattice dynamical systems, discrete time systems with continuous space, integro-differential equations, etc.), and to stimulate debates on open problems and future directions of research. In order to cover both physical and mathematical aspects, to avoid overlap between lectures and to appeal to a broad audience, 15 specialists were invited to present results on a given theme. By doing so, we were conscious of the fact that many significant contributors to the theory of CML could not present their results. But we had the feeling that a limited number of lectures could bring more material to a large audience than a series of talks.

This volume collects the notes written by the lecturers, sometimes with the help of collaborators. The themes cover numerical, theoretical and mathematical aspects of various spatially extended systems. More than the results themselves, concepts, techniques and tools developed for their analysis are presented. Since the investigation of a model on its own without any relationship to concrete situations has only little interest, examples of comparison and of

[2] Go to `http://nis-lab.is.s.u-tokyo.ac.jp/~nis/animation.html` to see the movie *Cloud simulation by CML* and to download the related paper [R. Miyazaki, S. Yoshida, Y. Dobashi and T. Nishita, Proc. of Pacific Graphics (2001) 363–372].

adaptation to physical and biological problems are given. The presentation is by no means complete, but we hope it can serve as a basis for future research on spatially extended systems.

Before going into details we present a schematic overview of most significant phenomena in Fig. 1. This picture collects the dynamical regimes, together with the transitions between regimes, which occur depending on the local map and on the interaction intensity. Naturally, the phenomenology here has been fairly simplified. It does not make any distinction between various forms of interaction (local coupling or global coupling). More importantly, it does not make any distinction between various spatially extended systems. To a smaller extent, neither does it take into account the lattice size dependence nor the dependence on boundary conditions.[3] Still we hope that this figure can guide the reader through the book.

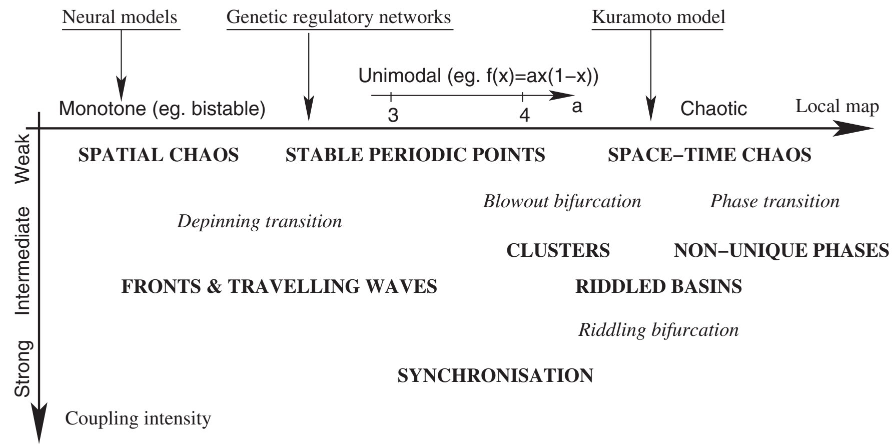

Fig. 1. Schematic representation of the phenomenology of CML (and of related models) *versus* the local map and the coupling intensity. Obviously, the phenomena may extend to larger domains than those indicated here. For instance, synchronisation may occur for any local map provided that the coupling (and the lattice size) is suitably chosen

The chapters have been assembled into 4 thematic parts. The first two parts are devoted to the description of statistical and geometric properties of CML. The third part collects results on the dynamics of monotone spatially extended systems. The last part concerns the introduction and analysis of models motivated by dynamical problems in Biology.

[3] Apart from an example in Sect. 5 of [Bunimovich], the effect of boundary conditions remain largely unravelled.

1 Statistical Properties of Coupled Chaotic Maps

Inspired by the analogy with spin models in Statistical Mechanics which emerges from symbolic dynamics, the consideration of global statistical properties of chaotic CML started soon afterward the introduction of CML, in 1988 precisely. The analogy suggests that, typically, phase transitions should occur when the coupling parameter increases. The transition is expected to split a unique space-time chaotic phase (high temperature) into several ordered phases (low temperature). However, the reputation of Statistical Mechanics technical difficulties warns that any attempt on a rigorous description of a phase transition in CML would face arduous problems.

To start with, characterising space-time chaotic phases is a problem in its own which has been the preliminary focus of many studies. A mixing hyperbolic dynamical system on a compact set has a (unique) natural phase with several equivalent characteristic properties. This equivalence fails in infinite lattices. Characterising the natural measures then needs to be addressed prior to any other statistical property in CML.

Various proposals have been made. Using again analogy with Statistical Mechanics, a natural measure should be the Gibbs measure of an appropriate Hamiltonian on the space-time lattice [Just and Schümser], [MacKay] and [Jarvenpää]. In the framework of the theory of Dynamical Systems and with an ergodic theorem in mind, a natural measure should describe the statistics of orbits issued from "typical" initial conditions, [Bunimovich] and [Keller and Liverani]. In the dual formulation of the dynamics, a natural measure should be the limit of iterations of any "regular" initial distribution, [Bunimovich], [MacKay] and [Keller and Liverani].

With a definition provided, the question of uniqueness of the natural measure in infinite lattices comes to the centre of attention. Contrary to the case of finite lattices of weakly coupled chaotic maps, requiring that all finite dimensional projections be absolutely continuous does not ensure uniqueness [Jarvenpää]. On another hand, due to infinite extension, some transients may last forever and can thus be defined as a proper phase; a phenomenon which does not exist in finite lattices [Just and Schmüser] and [MacKay]. Uniqueness can be shown however for small couplings in a suitable Banach space (of measures having finite dimensional marginals with at most exponentially growing total variation) [Keller and Liverani].

Two distinct approaches to phase transitions have been proposed. One approach is based on the formal derivation of a master equation for probabilities associated with atoms of the symbolic partition. It consists in showing that some transitions between atoms depend on the coupling parameter [Just and Schmüser]. However, this approach can be hardly controlled from a mathematical point of view. More importantly, changes in transition probabilities correspond to changes in the topology of the repeller (bifurcation) rather than to changes in its statistics only. Such changes may not be due to infinite spatial extension but may also occur in finite lattices. In this case, the term "phase transition" would not be appropriate [Bunimovich], [Just and Schmüser and [MacKay].

An alternative mathematically rigorous approach is to construct CML with prescribed phase transitions. The CML consist in piecewise affine mappings based on probabilistic cellular automata (PCA) which have been proved to possess a phase transition, in particular Toom's PCA [Just and Schmüser] and [MacKay]. Coupling there is introduced by letting the local map depend on symbolic states at neighbouring sites. Such models fairly differ from classical CML. However, this trick allows to overcome the unsolved problem of determining the symbolic dynamics of a CML for an arbitrary coupling parameter.

2 Geometric Aspects of Lattice Dynamical Systems

Beside focusing on specific phenomena such as phase transitions, and as suggested by the explicit dependence on the coupling parameter, a standard issue in CML is to describe the dynamics over the entire coupling parameter range. Due to competitions between local and interaction terms and between linear and nonlinear terms, this is a formidable task which has been accomplished only in particular cases. In arbitrary lattices, only the extreme regimes of weak and of strong couplings can be considered as satisfactorily described.

In view of perturbation theory, the dynamics at each site in a CML with weak coupling can be regarded as a local map perturbed by contributions from other sites. Accordingly, the behaviours in uncoupled and in weakly coupled regimes should be qualitatively the same provided that the local map dynamics is robust to perturbations.

The simplest case is when the local map has two stable fixed points. Then if the coupling parameter is small enough, just as in the uncoupled case the CML has an infinite set of stable fixed points on which the action of space translations has positive topological entropy [Afraimovich]. This property is called spatial chaos and extends to weakly coupled lattices of local maps with stable periodic orbits.

When the local map is strongly chaotic, space-time chaos exists for small coupling. That is to say, when the local map has a hyperbolic set with positive topological entropy, then the CML with sufficiently weak coupling has a hyperbolic set on which the $\mathbb{Z}^2$-action of space-time translations also has positive topological entropy.

In spite of being intuitively simple, weak interaction regimes gave the opportunity to adapt to lattice systems various dynamical systems techniques, e.g. persistence of uniform hyperbolicity under weak coupling, symbolic dynamics [Afraimovich] and [MacKay]. They also allowed to obtain results which are specific to lattice dynamical systems, e.g. description of space-time periodic configurations as orbits of a low-dimensional map, density of space-time (quasi-)periodic configurations with given (quasi-)period [Afraimovich].

Whereas structural stability of uncoupled systems does not depend on the symmetry of translation invariance and extends to some heterogeneous CML, in strongly coupled regimes, the dynamics relies on this symmetry.

The basic strongly coupled regime in translation invariant CML is synchronisation. In this context synchronisation means that the subset of constant configurations, namely the diagonal, attracts all orbits in phase space [Afraimovich] and [Maistrenko, Popovych and Tass].

Synchronisation takes place when all transverse eigenvalues of the mapping derivative computed at any point on the diagonal have modulus uniformly smaller than 1. The synchronisation is said to be chaotic if the tangential Lyapunov exponent on the diagonal (which in CML is nothing else but the Lyapunov exponent of the local map) is positive.

In the case where only the transverse Lyapunov exponents on the diagonal are negative (which happens when the coupling parameter decreases from the synchronisation regime) the basin of attraction of the diagonal is only local and may have a complex riddled structure, a phenomenon called partial synchronisation. Riddled basins are not limited to CML but emerge in a broader context, specifically in equivariant dynamical systems [Ashwin].

Riddled basins only concern a neighbourhood of the diagonal. The rest of phase space may contain orbits not asymptotically approaching the diagonal, a reminiscence of weakly coupled regimes. A simple example is a stable periodic orbit [Maistrenko, Popovych and Tass]. An example with a dense subset of unstable periodic orbits has also been exhibited [P. Glendinning, *Milnor attractors and topological attractors of a piecewise linear map*, Nonlinearity **14** (2001) 239–257].

3 Spatially Extended Systems with Monotone Dynamics

The typical situation for which the dynamics of a spatially extended system can be reasonably analysed over the whole coupling parameter range is that of systems with monotone dynamics. If an initial configuration lies below another initial configuration, then this ordering is preserved at later times.

Monotonicity is a classical property in parabolic partial differential equations (maximum principle). In lattices of coupled ordinary differential equations, it follows from cooperativity [Baesens]. For instance, it holds in the paradigmatic Frenkel-Kontorova model when assuming strong enough dissipation. In CML monotonicity holds for every $\epsilon \in [0, 1]$ provided that the local map f is an increasing function [Coutinho and Fernandez].

With the dynamics of chains in periodic potentials in mind, monotonicity can be completed with translation invariance and periodicity. Periodicity means that if the difference between two initial configurations equals, say 1 at all sites, then this difference remains unchanged at later times.

A monotone periodic translation invariant system has a regular and uniform nonlinear dynamics. Either each orbit remains sandwiched between stationary configurations (pinned regime) or all orbits indefinitely increase (or decrease) with finite velocity (sliding regime). In the sliding regime the propagation velocity is unique in phase space and there are corresponding travelling waves with rotationally ordered shape. The propagation velocity continuously depends on the system parameters. In particular, it is known to be positive for sufficiently large driving force (sufficiently large local map asymmetry in CML).

This phenomenology does not depend on the details of the model, a system either with continuous time [Baesens] or with discrete time. Neither it does depend on the details of the spatial interaction (discrete or continuous diffusion or both) [Coutinho and Fernandez]. This justifies substituting certain models by more convenient ones. In particular, one may assume the dynamics of lattice systems with small step sizes (discrete diffusion) to be suitably represented by the dynamics of a system defined on the whole real line (continuous diffusion), or vice-versa.

Excepted when generated by a driving force, transport may also be caused by a time dependent action on the system (non-autonomous system). In particular, switching on and off an asymmetric potential or switching on and off the diffusive interaction may also generate propagation (ratchet effect) [Floría, Baesens and Gómez-Gardeñes].

Even though the orbits remain bounded between two stationary configurations, there may be propagation. In this case, propagation concerns interfaces (discommensurations) between two contiguous stable stationary configurations. (Interfaces between a stable and an unstable configuration can also be relevant.) Bistable systems provide the basic framework where propagation of interfaces between stable phases can be analysed [Coutinho and Fernandez].

Bistable spatially extended systems satisfy monotonicity, translation invariance and the existence of two stable constant configurations at distance, say 1. In discrete time systems with arbitrary spatial interaction, the dynamics of interfaces is analogous to the previous one. There exists a unique asymptotic horizontal velocity for all interface orbits, this velocity depends continuously on the system parameters, and there are travelling waves (fronts).

4 Specific Lattice Dynamical Systems

In certain lattices with few sites, the dynamics can be described in the whole coupling range even though monotonicity is not assumed. A typical example with rich phenomenology is the Kuramoto model of globally coupled oscillators, a system of coupled ordinary differential equations. In this model, the sequence of bifurcations generated by decreasing the coupling is well-established [Maistrenko, Popovych and Tass]. Starting from a globally attracting synchronised orbit, bifurcations split asymptotic configurations into clusters with synchronised motions. For smaller couplings, clusters break into independent oscillators. In lattices with a large enough number of sites, this scenario includes a chaotic attractor for intermediate couplings. Motivated by synchronisation caused neurological diseases in brain function, the Kuramoto model has been employed to simulate the impact of a stimulation on an assembly of neurons.

The mechanisms leading to synchronisation in networks of neurons have been thoroughly investigated taking into account detailed neurons and synapses characteristics [Ermentrout]. In a different context where the neurons have a excitable dynamics and not an oscillatory one, propagating waves with regular or lurching motion have been exhibited. Some of these waves are the analogous of travelling fronts in systems with monotone dynamics mentioned above.

Another class of biological systems which comprehension involves network dynamics is that of genetic regulatory networks. The dynamical characteristics of the mechanisms involved in this context are a local piecewise contracting dynamics combined with a complex interaction graph [de Jong and Lima]. This combination is rather original in the framework of lattice dynamical systems and the resulting dynamics has only been completely described in networks with simple graphs.