International Journal of Bifurcation and Chaos, Vol. 18, No. 11 (2008) 3455–3459

GLOBAL SYNCHRONIZATION IN TRANSLATION INVARIANT COUPLED MAP LATTICES

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Received February 1, 2008; Revised March 18, 2008

A sufficient condition for global synchronization in coupled map lattices (CML) with translation invariant coupling and arbitrary individual map is proved. As in [Jost & Joy, 2001] where CML with reflection invariant couplings are considered, the condition only involves the linearized dynamics in the diagonal, namely for all points in the diagonal, the derivative must be contractive in all transverse directions. In addition to this result, a (weaker) condition that ensures the CML attractor to be composed of either 2-periodic or constant configurations, is also obtained.

Keywords: Global synchronization; coupled map lattices; translation invariant coupling.

Synchronization is probably the most commonly observed dynamical phenomenon in interacting (coupled) nonlinear systems. Generally speaking, it is said to occur in a multidimensional dynamical system when the attractor lies inside a one-dimensional subset of the phase space. The knowledge of the trajectory of a single coordinate allows to determine all coordinate trajectories [Boccaletti *et al.*, 2006; Pikovsky *et al.*, 2001]. In other words, all components asymptotically evolve "in phase", possibly in a chaotic motion [Pecora & Caroll, 1990]. In practice, the phenomenon takes various forms upon the system under consideration. For instance, phase synchronization takes place in systems of coupled oscillators [Fujisaka & Yamada, 1983] and master-slave, or generalized synchronization are possible phenomena in unidirectionally coupled systems [Hunt *et al.*, 1997; Rulkov *et al.*, 2001; Tresser *et al.*, 1995].

Another specific form of synchronization occurs in dynamical systems with symmetry where there is a convergence to the symmetry fixed point set (which is invariant under dynamics, see e.g. [Ashwin *et al.*, 1996]). Challenging problems in this framework then concern conditions on parameters (or on components) for synchronization. These conditions often involve transverse Lyapunov exponent in order to specify the basin of attraction of the symmetry fixed point set. In particular, a seminal result [Alexander *et al.*, 1992] states that this basin has positive Lebesgue measure (in phase space) when all transverse Lyapunov exponents are negative, for Lebesgue almost every point in the invariant set. Moreover, it is likely to have a fractal "riddled structure" (see the chapter by P. Ashwin in [Chazottes & Fernandez, 2005] for a mathematical definition), and actually presents such a structure in various examples. However, Ashwin *et al.* [1996] showed that this basin is indeed a neighborhood of the invariant set when the supremum of all transverse Lyapunov exponents, with respect to all ergodic measures supported in the invariant set, is negative.

Naturally, and as observed in [Ashwin *et al.*, 1996], without any further specification of the dynamics, these results are optimal and one cannot expect to control the global behavior. In some cases however, it is possible to determine the fate of every orbit in phase space from properties of the linearized dynamics in the invariant set.

CML have been introduced at the beginning of 1980s as simple discrete-time models of reaction–diffusion systems [Chazottes & Fernandez, 2005; Kaneko, 1993]. Their specificity resides in the definition of their mapping which is the composition of an individual map and a linear coupling. This presents the double advantage of being well-adapted to numerical simulations and to mathematical analysis. Recently [Lu & Chen, 2004] has established and analyzed synchronization conditions in CML with arbitrary linear coupling operators, mostly in the absence of symmetry but yet with invariant diagonal.

For reflection invariant coupled map lattices (CML), [Jost & Joy, 2001] proved that if all transverse eigenvalues of the jacobian matrix are contractive for all points in the diagonal (the invariant set in this case), then all points in phase space asymptotically approach the diagonal, a property called global synchronization. Although slightly stronger than the previous one, this condition is much simpler to check in practice.

In this Letter, we focus on translation invariant (but not necessarily reflection invariant) CML on periodic lattices. Translation invariance is usually assumed in most studies [Chazottes & Fernandez, 2005; Kaneko, 1993] as it reflects the simplifying assumptions that the individual systems are all identical and that the coupling is of diffusive type. As in [Jost & Joy, 2001], we show that global synchronization holds provided that all transverse directions are contractive for all points in the diagonal. Our condition is actually a bit more general than the one in [Jost & Joy, 2001] and the technique is different. The results, in particular, complete a previous result in [Lin *et al.*, 1999] on global synchronization for lattices of 2, 3 and 4 coupled logistic maps.

A translation invariant CML on the periodic lattice $\mathbb{Z}_L := \{s \in \mathbb{Z} \bmod L\}$ $(L > 1)$ is the dynamical system generated by the following induction relation in $\mathbb{R}$

$$x_s^{t+1} = \sum_{n\in\mathbb{Z}_L} c_n f(x_{s-n}^t), \quad s \in \mathbb{Z}_L. \tag{1}$$

Here the coefficients c_n are non-negative $(c_n \geq 0)$ and normalized $(\sum_{n\in\mathbb{Z}_L} c_n = 1)$. The most frequent examples are respectively, the asymmetric nearest neighbor coupling, the symmetric one $(L > 2)$ and the global coupling for which the coefficients are respectively given by $(\epsilon \in [0, 1])$

$$c_n = \begin{cases} 1-\epsilon & \text{if } n = 0 \\ \epsilon & \text{if } n = 1 \\ 0 & \text{otherwise} \end{cases}, \quad c_n = \begin{cases} 1-\epsilon & \text{if } n = 0 \\ \dfrac{\epsilon}{2} & \text{if } n = \pm 1 \\ 0 & \text{otherwise} \end{cases}$$

$$\text{and} \quad c_n = \begin{cases} 1-\epsilon & \text{if } n = 0 \\ \dfrac{\epsilon}{L} & \text{otherwise} \end{cases}$$

More generally, a coupling on the periodic lattice with L sites can be defined from any normalized sequence $\{\gamma_n\}_{n\in\mathbb{Z}}$ of non-negative coefficients by summing over the periods, namely

$$c_n = \sum_{k\in\mathbb{Z}} \gamma_{n+kL}.$$

The individual map f in (1) is assumed to possess an invariant interval I (which may be the whole $\mathbb{R}$) on which it satisfies the following inequality for some $K > 0$ (i.e. f is Lipschitz continuous)

$$|f(x) - f(y)| \leq K|x - y|, \quad \forall\, x, y \in I.$$

We assume that K is the smallest of such numbers. Note that if f is continuously differentiable, then K is the maximum of $|f'(x)|$ on I. However, f need not be differentiable for our purpose.

The CML (1) leaves invariant the diagonal in $I^{\mathbb{Z}_L}$. For any point $x_s = x$ $(s \in \mathbb{Z}_L)$ on this diagonal, the eigenvalues of the jacobian matrix associated with the CML derivative (which exists when f is differentiable) can be easily computed. Indeed, since this matrix commutes with translations on the lattice $\mathbb{Z}_L$, its eigenvectors in $\mathbb{C}^{\mathbb{Z}_L}$ are the elements e_k $(k = 0, \ldots, L-1)$ of the Fourier basis, where $(e_k)_s = (1/\sqrt{L})e^{\frac{2i\pi ks}{L}}$ $(s \in \mathbb{Z}_L)$. A simple calculation shows that the eigenvalues are given by $f'(x)\hat{c}_k$ where

$$\hat{c}_k = \sum_{n\in\mathbb{Z}_L} c_n e^{-\frac{2i\pi kn}{L}}.$$

In particular, $f'(x)\hat{c}_0 = f'(x)$ is the eigenvalue along the diagonal and all other eigenvalues correspond to transverse directions. The quantities $\hat{c}_k$ and $\hat{c}_{L-k}$ are complex conjugate for $k \neq 0, L/2$ (and equal in the case of reflection invariant couplings, i.e. $c_{-n} = c_n$). This implies that the contraction rate in the subset of $\mathbb{R}^{\mathbb{Z}_L}$ normal to the diagonal and defined by

$$\{ae_k + \bar{a}e_{L-k} : a \in \mathbb{C}\}$$

if $k \neq 0, L/2$ (and by $\{ae_{L/2} : a \in \mathbb{R}\}$ if $k = L/2$), is equal to $|\hat{c}_k|$. As a consequence, the normal Lyapunov exponents for any point on the diagonal, write $\lambda(x) + \log|\hat{c}_k|$ $(k = 1, \ldots, L-1)$ where $\lambda(x)$ is the individual map Lyapunov exponent evaluated at $x \in I$. According to Theorem 2.12 in [Ashwin *et al.*, 1996], the condition

$$\log(K) + \log \max_{1 \leq k \leq L-1} |\hat{c}_k| < 0$$

ensures the existence of a neighborhood of the diagonal in $I^{\mathbb{Z}_L}$ in which the orbit of every point asymptotically approaches the diagonal. Due to the specific structure of the CML (1), the very same condition actually implies that convergence to the diagonal holds for every point in phase space in such systems.

Theorem. *The condition*

$$K \max_{1 \leq k \leq L-1} |\hat{c}_k| < 1, \tag{2}$$

implies global synchronization of the CML. That is to say, the following limit holds

$$\lim_{t \to \infty} \max_{s \in \mathbb{Z}_L} |x_s^t - x_{s+1}^t| = 0$$

for all initial configurations $\{x_s^0\} \in I^{\mathbb{Z}_L}$.

(The proof is given below.) Several comments on this result can be made. First, since the definition of $\hat{c}_k$ implies $\max_{1 \leq k \leq L-1} |\hat{c}_k| \leq 1$, the condition (2) may hold in the case where $K > 1$ and in particular, for chaotic individual maps as it happens in the case of chaotic synchronization of CML. (On the other hand, global synchronization always hold when $K < 1$ because the whole CML is contracting in this case.)

Moreover, the coupling eigenvalues have the following limit (whose proof is given after the proof of the Theorem)

$$\lim_{L \to \infty} \max_{1 \leq k \leq L-1} |\hat{c}_k| = 1. \tag{3}$$

This implies that, for any coupling coefficient sequence $\{\gamma_n\}_{n \in \mathbb{Z}}$ and given individual map (with $K > 1$), the condition (2) becomes stronger and is hardly satisfied when the lattice size increases (and actually exceeds few sites in common examples). Alternatively, given a coupling coefficient sequence $\{\gamma_n\}_{n \in \mathbb{Z}}$ and a number of sites L, the condition (2) may not hold whenever K exceeds some threshold. For instance, for the symmetric nearest neighbor coupling defined above, $L = 2$ or 3 and $K > 1$, the condition (2) holds for all ϵ in some interval contained inside $[0, 1]$. However, if $L = 4$ (resp. $L = 6$), the same condition holds — for at least one value of ϵ in $[0, 1]$ — only if $K < 3$ (resp. $K < 5/3$).

Proof of the Theorem. The proof is not as trivial as the presentation may suggest. The natural strategy which consists in controlling the dynamics of transverse modes by applying the triangle inequality, does not work for $L > 2$. Indeed, this method provides upper bounds for both the modulus of transverse modes at time t and at time $t + 1$, but with no direct relationship between the two quantities. The alternative strategy which consists in directly estimating the difference $x_s^t - x_{s+1}^t$ does not work either when L exceeds a threshold, because it implies the whole spectrum and leads to the condition $K < 1$. Therefore, one has to find a better combination of coordinates in order to find the appropriate expression.

We consider the euclidean scalar product and associated norm $\|\cdot\|$ in $\mathbb{C}^{\mathbb{Z}_L}$ and the euclidean scalar product and associated norm $\|\cdot\|_{\mathbb{R}}$ in $\mathbb{R}^{\mathbb{Z}_L}$. Let $\mathcal{M}_1$ be the diagonal in $I^{\mathbb{Z}_L}$ and let $\mathcal{M}_1^{\perp} := \{x \in I^{\mathbb{Z}_L} : \sum_{s \in \mathbb{Z}_L} x_s = 0\}$ be its orthogonal complement. Let also C denote the coupling operator, i.e.

$$(C(x))_s = \sum_{n \in \mathbb{Z}_L} c_n x_{s-n}, \quad s \in \mathbb{Z}_L$$

where $x = \{x_s\}_{s \in \mathbb{Z}_L}$. In a first step, we show that global synchronization occurs under the condition $K \max_{x \in \mathcal{M}_1^{\perp} : \|x\|_{\mathbb{R}} = 1} \|C(x)\|_{\mathbb{R}} < 1$. Let R denote the (right) translation operator. Since the norms $\|\cdot\|_{\mathbb{R}}$ and $\max_{s \in \mathbb{Z}_L} |x_s|$ are equivalent, global synchronization is equivalent to the limit

$$\lim_{t \to \infty} \left\|x^t - R(x^t)\right\|_{\mathbb{R}} = 0,$$

for all $x^0 \in I^{\mathbb{Z}_L}$. The CML dynamics commutes with R and since $x - R(x) \in \mathcal{M}_1^{\perp}$ for all $x \in I^{\mathbb{Z}_L}$, we obtain the following inequality

$$\begin{aligned} &\left\|x^{t+1} - R(x^{t+1})\right\|_{\mathbb{R}} \\ &\quad \leq K \max_{x \in \mathcal{M}_1^{\perp} : \|x\|_{\mathbb{R}} = 1} \|C(x)\|_{\mathbb{R}} \|x^t - R(x^t)\|_{\mathbb{R}} \end{aligned}$$

An induction then completes the first step.

In a second step, we show that $\max_{x \in \mathcal{M}_1^{\perp} : \|x\|_{\mathbb{R}} = 1} \|C(x)\|_{\mathbb{R}} = \max_{1 \leq k \leq L-1} |\hat{c}_k|$. We identify every vector $\{x_s\} \in \mathbb{R}^{\mathbb{Z}_L}$ with the vector $\{z_s\} \in \mathbb{C}^{\mathbb{Z}_L}$ where $z_s = x_s$ for all s. It follows that every vector $x \in \mathcal{M}_1^\perp$ can be viewed as belonging to the subspace orthogonal to the diagonal in $\mathbb{C}^{\mathbb{Z}_L}$. Therefore, this vector writes $x = \sum_{k=1}^{L-1} x_k e_k$ with complex coordinates x_k. Since e_k are the eigenvectors of C in $\mathbb{C}^{\mathbb{Z}_L}$ with eigenvalues $\hat{c}_k$ we have

$$C(x) = \sum_{k=1}^{L-1} x_k \hat{c}_k e_k,$$

and then $\|C(x)\| \leq \max_{1\leq k\leq L-1} |\hat{c}_k| \|x\|$. Since $C(x) \in \mathbb{R}^{\mathbb{Z}_L}$ and $\|x\| = \|x\|_{\mathbb{R}}$, we have actually showed that

$$\max_{x\in\mathcal{M}_1^\perp : \|x\|_{\mathbb{R}}=1} \|C(x)\|_{\mathbb{R}} \leq \max_{1\leq k\leq L-1} |\hat{c}_k| .$$

It only remains to show that the inequality is indeed an equality. Let $k' \in \{1, \ldots, L-1\}$ be such that

$$|\hat{c}_{k'}| = \max_{1\leq k\leq L-1} |\hat{c}_k| .$$

The normalized vector $v = (1/\sqrt{2})(e_{k'} + e_{L-k'})$ (resp. $v = e_{L/2}$ if $k' = L/2$) has real coordinates that belong to $\mathcal{M}_1^\perp$ and satisfy $\|C(v)\| = |\hat{c}_{k'}|$ since $\hat{c}_{k'}$ and $\hat{c}_{L-k'}$ are complex conjugate. In other words, there exists a normalized vector $v \in \mathcal{M}_1^\perp$ such that $\|C(v)\|_{\mathbb{R}} = \max_{1\leq k\leq L-1} |\hat{c}_k|$. The theorem is proved.

Proof of relation (3). Firstly, the triangle inequality implies that for every $n' > 0$ we have

$$\left|\sum_{n\in\mathbb{Z}} \gamma_n e^{-\frac{2i\pi n}{L}}\right| \geq \left|\sum_{|n|\leq n'} \gamma_n e^{-\frac{2i\pi n}{L}}\right| - \left|\sum_{|n|>n'} \gamma_n e^{-\frac{2i\pi n}{L}}\right| .$$

The sequence $\{\gamma_n\}$ is assumed to be summable. Thus for every $\delta > 0$, there exists n_δ such that

$$\left|\sum_{|n|>n_\delta} \gamma_n e^{-\frac{2i\pi n}{L}}\right| \leq \sum_{|n|>n_\delta} \gamma_n < \delta.$$

Now, let $\delta < 1$ and define L_δ to be sufficiently large so that for all $L \geq L_\delta$, we have $\cos(2\pi n/L) \geq 1 - \delta$ for all $|n| \leq n_\delta$. The two estimates imply that the following inequality holds for $L \geq L_\delta$

$$\left|\sum_{|n|\leq n_\delta} \gamma_n e^{-\frac{2i\pi n}{L}}\right| \geq \left|\sum_{|n|\leq n_\delta} \gamma_n \cos\left(\frac{2\pi n}{L}\right)\right| \geq (1-\delta) \sum_{|n|\leq n_\delta} \gamma_n \geq (1-\delta)^2$$

It results that $|\hat{c}_1| \geq (1-\delta)^2 - \delta$ for $L \geq L_\delta$ and the desired limit (3) follows. ∎

To conclude this Letter, we mention that global synchronization can be seen as a special case of convergence to a spatially periodic configuration subset with prescribed period, namely period 1 in the present case. The arguments of the proof above can be adapted in order to obtain a condition such that the CML attractor is composed of periodic configurations with (not necessarily minimal) period, a given divisor of L. We only consider here the situation where this period is 2 and we provide the corresponding condition.

Proposition. *Assume that L is even and that the following condition holds*

$$K \max_{1\leq k\leq L-1 : k\neq L/2} |\hat{c}_k| < 1. \tag{4}$$

Then for any initial configuration $\{x_s^0\} \in I^{\mathbb{Z}_L}$, we have

$$\lim_{t\to+\infty} \max_{s\in\mathbb{Z}_L} |x_s^t - x_{s+2}^t| = 0.$$

Proof. We start by observing that for all $x \in I^{\mathbb{Z}_L}$, we have $x - R^2(x) \in \mathcal{M}_2^\perp$ where

$$\mathcal{M}_2^\perp = \left\{x \in I^{\mathbb{Z}_L} : \sum_{s\in\mathbb{Z}_L} x_s = 0 \right.$$
$$\left. \text{and} \sum_{s\in\mathbb{Z}_L} (-1)^s x_s = 0\right\}$$

is the hyperplane orthogonal both to the diagonal and to the linear subspace generated by $e_{L/2}$. As in the proof of the Theorem, we conclude that the condition $K \max_{x\in\mathcal{M}_2^\perp : \|x\|_{\mathbb{R}}=1} \|C(x)\|_{\mathbb{R}} < 1$ implies the limit $\lim_{t\to+\infty} \left\|x^t - R^2(x^t)\right\|_{\mathbb{R}} = 0$ which is equivalent to the desired result. Now, by definition of $\mathcal{M}_2^\perp$, every configuration $x \in \mathcal{M}_2^\perp$ (viewed as an element of $\mathbb{C}^{\mathbb{Z}_L}$) writes $x = \sum_{1\leq k\leq L-1,\, k\neq L/2} x_k e_k$. As in the proof of the theorem, this implies that

$$\max_{x\in\mathcal{M}_2^\perp : \|x\|_{\mathbb{R}}=1} \|C(x)\|_{\mathbb{R}} = \max_{1\leq k\leq L-1 : k\neq L/2} |\hat{c}_k|$$

The proposition then easily follows. ∎

Acknowledgments

I warmly thank V. Afraimovich for stimulating and fruitful discussions. I am also indebted to P. Guiraud for a careful reading of the manuscript which allowed to improve the presentation.

References

Alexander, J., Kan, I., Yorke, J. & You, Z. [1992] "Riddled basins," *Int. J. Bifurcation and Chaos* **2**, 795–813.

Ashwin, P., Buescu, J. & Stewart, I. [1996] "From attractor to chaotic saddle: A tale of transverse instability," *Nonlinearity* **9**, 703–737.

Bocaletti, S., Latora, V., Moreno, Y., Chavez, M. & Wang, D.-U. [2006] "Complex networks: Structure and dynamics," *Phys. Rep.* **424**, 175–308.

Chazottes, J.-R. & Fernandez, B. (ed.) [2005] *Dynamics of Coupled Map Lattices and of Related Spatially Extended Systems*, Lecture Notes in Physics, Vol. 671 (Springer).

Fujisaka, H. & Yamada, T. [1983] "Stability theory of synchronized motion in coupled oscillator systems," *Prog. Theoret. Phys.* **69**, 32–47.

Hunt, B., Ott, E. & Yorke, J. [1997] "Differentiable generalized synchronization of chaos," *Phys. Rev. E* **55**, 4029–4034.

Jost, J. & Joy, M. P. [2001] "Spectral properties and synchronization in coupled map lattices," *Phys. Rev. E* **65**, 016201.

Kaneko, K. (ed.) [1993] *Theory and Applications of Coupled Map Lattices* (Wiley).

Lin, W.-W., Peng, C.-C. & Wang, C.-S. [1999] "Synchronisation in coupled map lattices with periodic boundary condition," *Int. J. Bifurcation and Chaos* **9**, 1635–1652.

Lu, W. & Chen, T. [2004] "Synchronization analysis of linearly coupled networks of discrete time systems," *Physica D* **198**, 148–168.

Pecora, L. & Caroll, T. [1990] "Synchronisation of chaotic systems," *Phys. Rev. Lett.* **64**, p. 821.

Pikovsky, A., Rosenblum, M. & Kurths, J. [2001] *Synchronization. A Universal Concept in Nonlinear Sciences* (Cambridge University Press).

Rulkov, N., Afraimovich, V., Lewis, C., Chazottes, J.-R. & Cordonet, A. [2001] "Multivalued mappings in generalized chaos synchronization," *Phys. Rev. E* **64**, 016217.

Tresser, C., Wolfork, P. & Bass, H. [1995] "Master–slave synchronization from the point of view of global dynamics," *Chaos* **5**, 693–699.