Spatially Extended Monotone Mappings

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1 Introduction

This chapter deals with the study of travelling waves in discrete time spatially extended systems with monotone dynamics. Such systems appear for instance in alloy solidification, in population dynamics and in solid-state physics. Special emphasis is made on the existence of travelling waves, on the uniqueness of their velocity and on their relevance for the description of propagation phenomena in such systems.

The first section deals with interfaces between two stable homogeneous phases and their propagation in the form of fronts. The analysis applies to systems of bistable one-dimensional maps coupled via the convolution with an arbitrary distribution function [6]. This analysis completes a previous work on piecewise affine bistable CML [3, 4] and its extension to systems of piecewise affine one-dimensional maps coupled via convolution [5].

The second section deals with travelling waves in monotonous extended systems driven by spatially periodic forces. These systems are inspired by discrete time analogues of the dissipative dynamics of Frenkel-Kontorova models (see the chapters by Floría, Baesens and Gómez-Gardeñes and by Baesens for such dynamics in continuous time). For such nonlinear systems, a dispersion relation is obtained and the existence of travelling waves with arbitrary wave and corresponding frequency is shown [7].

In spite of similarities with other works in the literature (see e.g. [12]) the methods and, particularly, the formalism developed in the papers [3, 4, 5, 6, 7] are quite distinct and original. They encompass in a unified framework, systems with continuous couplings and systems with discrete couplings. In particular, changes in the dynamics of travelling waves (e.g. changes in shape and in velocity) are described when the coupling continuously changes from a discrete to a continuous one.

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2 Bistable Extended Maps

2.1 From Bistable CML to General Bistable Extended Mappings

As a starting point, we consider the basic model of CML

$$u_s^{t+1} = (1-\varepsilon)f(u_s^t) + \frac{\varepsilon}{2}\left(f(u_{s-1}^t) + f(u_{s+1}^t)\right) \tag{1}$$

where the real number $u_s^t \in [0,1]$ represents the element of a lattice configuration at discrete time $t \in \mathbb{Z}$ and discrete space $s \in \mathbb{Z}$, and where $\varepsilon \in [0,1)$ is the coupling strength. The map f is a **bistable map** from $[0,1]$ into itself. A bistable map is a continuous increasing mapping from $[0,1]$ into itself with exactly 3 fixed points, namely the points 0, c and 1. The points 0 and 1 are attracting and c is unstable, see Fig. 1.

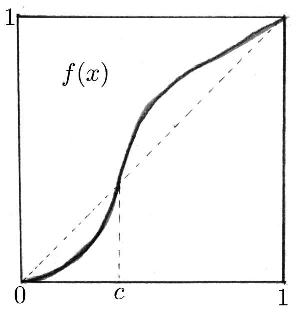

Fig. 1. A bistable map f

The evolution of a configuration under a bistable CML can be viewed as a local force which impels convergence to some steady state (a stable fixed point of f, either 0 or 1), followed by a diffusion process which takes the form of the following linear operator

$$(\mathrm{L}u)_s = (1-\varepsilon)u_s + \frac{\varepsilon}{2}(u_{s+1} + u_{s-1}) \ . \tag{2}$$

In particular, if at time t the configuration satisfies $u_s^t > c$ for all $s \in \mathbb{Z}$, then the evolution forces the configuration to converge uniformly to the fixed point $u_s = 1$ for all $s \in \mathbb{Z}$. On the opposite, if $u_s^t < c$ for all $s \in \mathbb{Z}$, then the evolution forces the configuration to converge uniformly to the fixed point $u_s = 0$ for all $s \in \mathbb{Z}$. Hence these fixed points represent stable phases of the dynamics.

The goal of this first section is to investigate the competition between these two phases, namely the evolution of configurations which at some time satisfy $u_s^t < c$ for sufficiently large negative s and $u_s^t > c$ for sufficiently large positive s. Typically, the resulting motion is the invasion of one phase into the other one, an invasion ruled by special solutions of (1), namely the fronts.

A front of rational velocity $\frac{p}{q}$ for the CML (1) is a configuration which satisfies this evolution equation and the relations

$$\forall s, t : u_s^{t+q} = u_{s-p}^t, \quad \lim_{s\to-\infty} u_s^t = 0 \quad \text{and} \quad \lim_{s\to+\infty} u_s^t = 1 \ .$$

Before starting the investigation of such fronts, one may wonder about the existence of fronts with irrational velocity in CML. A simple way to take into account both rational and irrational velocities in a unique formalism is to define a **front** of velocity v for the CML (1) as a configuration which satisfies

$$u_s^t = \phi\,(s - tv)$$

where $\phi : \mathbb{R} \to [0,1]$ is a real function, called the **front shape**, which satisfies

$$\lim_{x\to-\infty} \phi\,(x) = 0 \quad \text{and} \quad \lim_{x\to+\infty} \phi\,(x) = 1 \ .$$

In other words the introduction of functions of real variable in CML allows to study fronts with irrational velocities. However, using also this formalism in the description of fronts with rational velocities has proved to be useful for the global comprehension of fronts dynamics.

Early studies in this direction showed the existence of such fronts (of rational or irrational velocity depending on parameters) for (1) when f is a discontinuous piecewise affine map with constant slope [3]. The existence of fronts has also been shown in a CML with continuous piecewise affine bistable local map and unidirectional coupling [1]. The proof in [3] is achieved by using an explicit construction of the front shape for this CML. In order to show the existence of such fronts when the local map f is an arbitrary (continuous) bistable map, it is useful to fully generalise the model under consideration.

Actually, one may not restrict oneself to CML defined by (1) but also consider more general CML whose iterations involve larger, even unbounded, neighbourhoods. In addition, a reversal symmetric coupling is not required and this assumption can be dropped. That is to say, one may consider the following coupling operator

$$(\mathrm{L}u)_s = \sum_{n\in\mathbb{Z}} \ell_n u_{s-n}$$

where the coefficients ℓ_n are nonnegative real numbers such that

$$\sum_{n\in\mathbb{Z}} \ell_n = 1 \ .$$

Note that the coupling operator in (2) is recovered by choosing $\ell_{-1} = \ell_1 = \frac{\varepsilon}{2}$, $\ell_0 = 1 - \varepsilon$ and $\ell_n = 0$ for $n \notin \{-1, 0, 1\}$.

Since the front shape is defined as a function of real variable, it is natural to extend the action of the dynamics to the functions of real variable. That is to say instead of the CML (1), we consider

$$u^{t+1}(x) = \sum_{n\in\mathbb{Z}} \ell_n f(u^t(x-n)) \ .$$

where each u^t is a function from $\mathbb{R}$ to $[0,1]$.

Of course, the dynamics of lattice configurations is recovered by considering the invariant set of functions which are constant on every interval $[s, s+1)$ where $s \in \mathbb{Z}$. But such an extension provides an appropriate framework to the front shape dynamics.

At this stage, an additional extension appears immediately. One may consider diffusive linear operators of the form

$$(\mathrm{L}u)(x) = \sum_{n\in\mathbb{Z}} \ell_n u(x - r_n)$$

where the coefficients ℓ_n are nonnegative real numbers such that $\sum_{n\in\mathbb{Z}} \ell_n = 1$ and r_n are arbitrary real numbers (not only integers). An alternative way of writing this operator is by using the **convolution** with a distribution function h. Recall that such a convolution is defined by the Lebesgue-Stieltjes integral

$$(\mathrm{L}u)(x) = h * u(x) := \int_{\mathbb{R}} u(x-y)dh(y) \ . \tag{3}$$

That is to say, one may consider diffusive linear operator of the form $Lu = h*u$. In the previous case, the coupling operator can be written $h * u$ with the distribution function h being given by $h(x) = \sum_{n\in\mathbb{Z}} \ell_n H(x - r_n)$ where H is the Heaviside function

$$H(x) = \begin{cases} 0 & \text{if} \quad x < 0 \\ 1 & \text{if} \quad 0 \le x \end{cases}$$

Convolutions are not limited to discrete distribution functions and, as a final extension, one may consider $Lu = h * u$ where h is an arbitrary distribution function, that is to say, any increasing function with the following limits $\lim_{x\to-\infty} h(x) = 0$ and $\lim_{x\to+\infty} h(x) = 1$.

Therefore instead of only considering the dynamics of fronts in bistable CML, we consider the dynamics of fronts in bistable (spatially) extended maps whose iterations write

$$u^{t+1} = Fu^t = h * f \circ u^t \tag{4}$$

where h is an arbitrary distribution function and f an arbitrary bistable map. (A further extension will be considered in Sect. 2.4.)

In order to have a well-defined convolution operator (3) for arbitrary distribution function h, the functions u need be Borel measurable. Accordingly we consider the dynamics (4) in the set $\mathcal{B}$ of Borel-measurable functions defined on $\mathbb{R}$ with values in $[0,1]$.

It is noteworthy that the present formalism collects in a unique framework, CML and classical models with continuous diffusive couplings. On one hand the basic model of CML is recovered for

$$h(x) = (1-\varepsilon)H(x) + \frac{\varepsilon}{2}(H(x+1) + H(x-1)) \ .$$

On the other hand by choosing h to be the absolutely continuous distribution function with heat kernel

$$h(x) = \int_{-\infty}^{x} e^{-\pi y^2} dy, \quad x \in \mathbb{R}$$

the map $Fu(x) = (h * f \circ u)(x) = \int_{\mathbb{R}} f \circ u(x-y) e^{-\pi y^2} dy$ gives an integral formulation of a reaction-diffusion process in discrete time. Indeed, we then have $Fu(x) = w_u(x, 1/4\pi)$, where $w_u(x,t)$ is the solution of the initial value problem $\frac{\partial w_u}{\partial t} = \frac{\partial^2 w_u}{\partial x^2}$ and $w_u\,(x,0) = f \circ u(x)$.

The rest of this section is dedicated to a sketch of the front analysis in bistable extended maps defined by (4). This amounts to prove the existence of a velocity v and of a front shape ϕ which solves the front equation

$$\phi\,(x-v) = h * f \circ \phi\,(x) \ .$$

2.2 Basic Properties

Every bistable extended map F defined by (4) has three basic properties. The first property, which is intensively used in the analysis, is **homogeneity**. Homogeneity is expressed by the relation

$$T^v F = F T^v \quad \text{for all } v \in \mathbb{R} \tag{5}$$

where T^v is the **translation** by v, namely the operator acting in $\mathcal{B}$ and defined by $T^v u(x) = u(x-v)$ for all $x \in \mathbb{R}$.

The second important property of F is **continuity** in the sense of pointwise convergence, namely

$$\forall x \in \mathbb{R} \quad \lim_{n\to\infty} u_n\,(x) = u\,(x) \quad \Rightarrow \quad \forall x \in \mathbb{R} \quad \lim_{n\to\infty} Fu_n\,(x) = Fu\,(x) \ . \tag{6}$$

Finally, the third fundamental property is **monotonicity**, namely

$$u \le v \quad \Rightarrow \quad Fu \le Fv \ . \tag{7}$$

Using these three properties we can deduce other important facts:

(a) Under the action of F, every increasing function is mapped into an increasing function. Using the fact a function is increasing iff it lies above any of its right translation, we have

$$\forall \delta > 0 \quad T^\delta u \le u \quad \Rightarrow \quad T^\delta F u \le F u \ .$$

(b) The map F commutes with the **projection** P_ℓ on left continuous functions. Indeed this projection is defined by $P_\ell u(x) = \lim_{\substack{y\to x\\ y<x}} u(y)$ and for any function in its domain, we have $P_\ell u(x) = \lim_{n\to\infty} u(x - \frac{1}{n})$. Using homogeneity and continuity, we conclude that $F P_\ell u = P_\ell F u$. Similarly, one shows that F commutes with the projection P_r on right continuous functions.

2.3 Results and Concepts

Existence of Fronts

As suggested before, any bistable extended map of the form (4) has fronts for some velocity v. This is formally claimed in the following statement.

Theorem 2.1. *For any distribution function h and any bistable map f, there exists a velocity $v \in \mathbb{R}$ and an increasing function ϕ with the following limits $\lim_{x\to-\infty} \phi(x) = 0$ and $\lim_{x\to+\infty} \phi(x) = 1$ which solves the front equation $T^v\phi = h * f \circ \phi$.*

The proof of this Theorem is sketched in Sect. 2.5. The proof is accomplished by a construction of an increasing function ϕ with the desired properties for a chosen velocity v.

Note that monotonicity of the front shape ϕ is not imposed by the front equation. Indeed in some cases (e.g. in weakly coupled CML or in strongly unidirectionally coupled CML), it may happen that a bistable extended map also has fronts with non monotonous shape.

Moreover, uniqueness of monotonous shape cannot be expected in general. There are examples of bistable CML with several monotone front shapes which cannot be identified by applying translations.

Bistable Regular Maps and the Uniqueness of Front Velocity

In spite of the front shape need not be unique, one may wonder about the uniqueness of the velocity. It turns out that this uniqueness holds provided that the map f is so-called regular, a fairly general situation. Indeed, a bistable map f is said to be **regular** if it is a weak contraction in the neighbourhoods of the stable fixed points (see Fig. 2), i.e. if we have

$$\exists\delta > 0 \quad [x, y \in (0, \delta) \quad \text{or} \quad x, y \in (1-\delta, 1)] \quad \Rightarrow |f(x) - f(y)| \leq |x - y| \ .$$

That a bistable map be regular is a quite mild condition relies on Taylor expansion. Indeed by using Taylor formula, one obtains the following sufficient conditions for a bistable map f to be regular

f is analytic, or
$f \in C^1$ and $f'(0) < 1$ and $f'(1) < 1$, or
$f \in C^2$ and $f''(0) \neq 0$ and $f''(1) \neq 0$, or
$f \in C^3$ and $f'''(0) \neq 0$ and $f'''(1) \neq 0$, and
...

Nevertheless, one can exhibit examples of non regular C^∞ bistable maps.

With regularity provided, Theorem 2.1 can be completed by an assertion on velocity uniqueness.

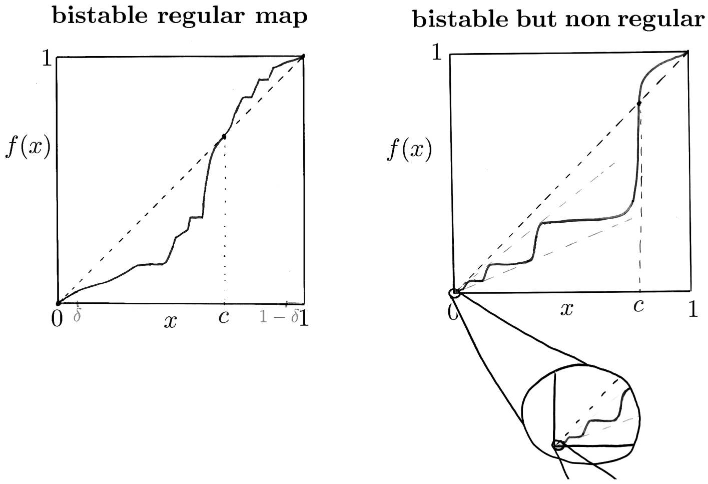

Fig. 2. Bistable regular and non regular local maps

Theorem 2.2. *For any distribution function h and any* regular *bistable map f, there exists a* unique *velocity $v \in \mathbb{R}$ and an increasing function ϕ with the following limits* $\lim_{x\to-\infty} \phi(x) = 0$ *and* $\lim_{x\to+\infty} \phi(x) = 1$ *which solves the front equation $T^v \phi = h * f \circ \phi$.*

Hausdorff Distance of Increasing Functions

Uniqueness of the front velocity in extended systems with regular bistable maps naturally addresses the question of the dependence of this velocity on the local map and on the coupling operator, i.e. the dependence of $v(f, h)$ on f and on h.

The dependence under consideration here is continuity with changes in f and in h. In order to address this problem, adequate notions of convergence both for local maps f and for distribution functions h need to be introduced.

As far as local maps are concerned, convergence is understood in the pointwise sense, i.e. $\{f_n\}$ converges to f iff $\lim_{n\to\infty} f_n(x) = f(x)$ for all $x \in [0, 1]$.

As far as distribution functions are concerned, the convergence is ruled by a distance in the set of (right continuous) increasing functions. The distance between two distribution functions h and h' is given by

$$d(h, h') = \inf\{\varepsilon > 0 \ : \ h(x - \varepsilon) - \varepsilon \le h'(x) \le h(x + \varepsilon) + \varepsilon, \quad \forall x \in \mathbb{R} \ \} \ . \quad (8)$$

For this distance, the ball of radius ε centred at h is the set of functions for which the graph lies in the band of width $2\sqrt{2}\varepsilon$ in the direction of the line $y = -x$ around the graph Gh of h. The graph of h is defined by

$$Gh = \{(x, y) : P_\ell h\,(x) \le y \le P_r h\,(x)\} \ .$$

In fact it is not difficult to show that this distance is the Hausdorff distance restricted to graphs of such functions:

$$d(h,h') = \max\left\{\sup_{z_1\in Gh}\left(\inf_{z_2\in Gh'}\|z_1-z_2\|\right), \sup_{z_1\in Gh'}\left(\inf_{z_2\in Gh}\|z_1-z_2\|\right)\right\}$$

where the $\mathbb{R}^2$ norm $\|.\|$ is given by $\|(x,y)\| = \max\{|x|,|y|\}$.

By using relation (8), one shows that the convergence with respect to the distance $d(\cdot,\cdot)$ is equivalent to the convergence at all continuity points, the usual convergence of distribution functions [10]. Precisely, we have $\lim_{n\to\infty} d(h_n,h) = 0$ iff $\lim_{n\to\infty} h_n(x) = h(x)$ for all x where h is continuous [6].

The advantage of such a distance is that it allows continuous distribution functions to converge to discontinuous ones and vice-versa. As a particular consequence, changes in front velocity can be analysed when continuously passing from a model with continuous diffusive operator to a model with discrete diffusive operator (and vice-versa).

Continuity of the Front Velocity

The continuous dependence of the front velocity with respect both to the local map and to the distribution function is given by the following statement. Assume that f is regular and let $v(f,h)$ be the unique front velocity of the mapping $Fu = h * f \circ u$.

Theorem 2.3. *Let $\{f_n\}_{n\in\mathbb{N}}$ be a sequence of regular bistable maps which converges pointwise to a bistable regular map f. Let $\{h_n\}_{n\in\mathbb{N}}$ be a sequence of distribution functions and h be a distribution function such that $\lim\limits_{n\to\infty} d(h_n,h) = 0$. Then $\lim\limits_{n\to\infty} v(f_n,h_n) = v(f,h)$.*

In particular, the front velocity varies continuously with any parameter of the local map (provided that the map pointwise depends continuously on its parameter(s)) and with any coupling parameter (provided the distribution function depends continuously on this parameter). For instance, in the CML defined by (1), the front velocity depends continuously on ε.

A special consequence of this result is the existence of fronts with irrational velocity (for appropriate value of ϵ) in any CML (1) for which the front velocity is not constant when ε moves in $[0,1]$.

As suggested before, Theorem 2.3 contains the claim that the front velocity of an extended bistable map with infinite range coupling can be approximated to arbitrary accuracy by the front velocity of an extended bistable map with finite range discrete coupling, and vice-versa.

Interfaces and Reference Centres $J_a(\psi)$

Once the existence of front has been established, the natural question to address is their Lyapunov stability. Lyapunov stability of fronts is an elaborated question which lies beyond the scope of this chapter.

In this section we provide some information about the dynamics of configurations which need not be monotone, nor need to cross once the unstable fixed point c of f. These configurations are called interfaces. An **interface** is a function $u \in \mathcal{B}$ such that there exists $c_- \in (0,c)$, $c_+ \in (c,1)$ and $j_1 \leq j_2 \in \mathbb{R}$ so that $u(x) \leq c_-$ if $x \leq j_1$ and $u(x) \geq c_+$ if $x \geq j_2$ (see Fig. 3 for an example of an interface crossing several times the point c.)

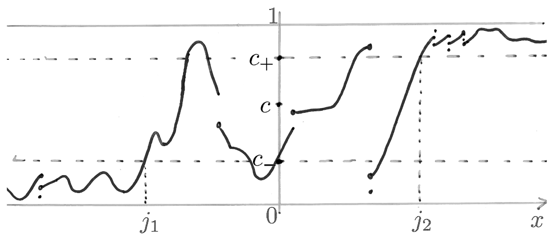

Fig. 3. An interface function u

Interfaces possess the following dynamical properties. If u is an interface, then every iteration $F^t u$ $(t \geq 0)$ is an interface. Moreover, the numbers c_- (resp. c_+) can be chosen arbitrarily near to 0 (resp. 1) provided that t is chosen (accordingly) large enough.

As shown below, the asymptotic dynamical property shared by all interfaces is a unique propagation velocity, the front velocity of course.

In order to compute this velocity, the interface location at each time is measured according to a reference threshold a. We introduce the **reference centre** of an interface u as the smallest point at which the function is not smaller than a, i.e.

$$J_a(u) = \inf \{x \in \mathbb{R} \ : \ u(x) \geq a\} \ .$$

In the case where this quantity is finite, by applying a translation, the function can be centred at 0. Indeed, we have $J_a(T^{-J_a(u)}u) = 0$.

Velocity of Interfaces

According to the previous dynamical property, for any interface and any $a \in (0,1)$, the quantity $J_a(F^t u)$ is finite for all t sufficiently large. The next statement claims that any interface has asymptotically the front velocity, no matter the initial number of crossing the level c is.

Theorem 2.4. *Let h be a distribution function and let f be a regular bistable map. For every interface u and every $a \in (0,1)$, we have*

$$\lim_{t\to+\infty} \frac{J_a(F^t u)}{t} = v(f,h) \ .$$

Needless to say that the front velocity is an important characteristic of extended bistable map. It plays a similar role to the one played by the rotation number in circle maps.

2.4 Generalisation

The results on front dynamics extend to linear convex combinations of maps of the form $Fu = h * f \circ u$. An interesting application of such an extension resides in lattice dynamical systems as introduced in the chapter by Bunimovich in this volume.

Instead of the map F defined by (4) we now consider the map F defined by

$$Fu = \sum_{k\in\mathbb{N}} a_k h_k * f_k \circ u, \quad u \in \mathcal{B} \ .$$

Here the numbers $a_k \geq 0$ and $\sum_{k\in\mathbb{N}} a_k = 1$. The functions h_k are distributions functions and the maps f_k are continuous increasing maps defined on $[0,1]$ such that there exists $c \in (0,1)$ so that for every $k \in \mathbb{N}$ we have

$$f_k(x) \leq x \quad \text{if } 0 \leq x \leq c \quad \text{and} \quad x \leq f_k(x) \quad \text{if } c \leq x \leq 1 \ .$$

Moreover, we assume that the map

$$f = \sum_{k\in\mathbb{N}} a_k f_k$$

is bistable. Its unstable fixed point is then c.

In addition, we say that the map F is regular if there exists $\delta > 0$ such that for every $k \in \mathbb{N}$ we have

$$|f_k(x) - f_k(y)| \leq |x-y| \quad \text{if } x,y \in (0,\delta) \quad \text{or if } x,y \in (1-\delta,1) \ .$$

All previous results on existence of fronts (Theorem 2.1), uniqueness of the velocity (Theorem 2.2), continuous dependence of the velocity on the parameters (Theorem 2.3) and existence and uniqueness of the velocity of interfaces (Theorem 4) extend to the present mapping F.

Example. Lattice dynamical system. Let $\varepsilon \in (0,1)$ and f be a regular bistable map such that the map f_0 defined on $[0,1]$ by $f_0(x) = \frac{f(x)-\varepsilon x}{1-\varepsilon}$ is increasing. Let $f_1(x) = x$, $a_0 = 1-\varepsilon$, $a_1 = \varepsilon$, $a_k = 0$ if $k > 1$, $h_0 = H$ and $h_1 = \frac{1}{2}(T^1 H + T^{-1} H)$.

The map

$$Fu(x) = \sum_{k\in\mathbb{N}} a_k h_k * f_k \circ u(x) = f \circ u(x) + \frac{\varepsilon}{2}(u(x-1) - 2u(x) + u(x+1))$$

satisfies the desired properties and is regular.

2.5 Sketch of the Proof of Existence of Fronts

This section presents a brief description of the proof of front existence. The complete proof is given in [6].

The first step consists in introducing **subfronts**, that is to say increasing functions which satisfy the inequality $Fu \leq T^v u$. To be precise, let $\mathcal{I} \subset \mathcal{B}$ be the subset composed of increasing functions, let $v \in \mathbb{R}$ and $c_+ \in (c, 1)$. Consider the set of subfronts of velocity v defined by

$$\mathcal{S}_{v,c_+} = \left\{ \psi \in \mathcal{I} \ : \ F\psi \leq T^v \psi \text{ and } J_{c_+}(\psi) = 0 \right\} \ .$$

If $\mathcal{S}_{v,c_+}$ is not empty, consider the function

$$\eta_v(x) = \inf_{u \in S_{v,c+}} u(x), \quad x \in \mathbb{R} \ .$$

It turns out that $\eta_v \in \mathcal{S}_{v,c_+}$ and therefore η_v is a minimal sub-front of velocity v.

In a second step, we consider the maximal sub-fronts velocity

$$\bar{v} = \sup \left\{ v \in \mathbb{R} \ : \ \mathcal{S}_{v,c_+} \neq \emptyset \right\}$$

and we consider the minimal sub-front of maximal velocity, namely $\eta_{\bar{v}}$. (This minimal subfront exists because one shows that $S_{\bar{v},c_+} \neq \emptyset$.) The construction suggests $\eta_{\bar{v}}$ is a good candidate to solving the front equation. However, this is not always the case.

In order to construct a front shape from this function, we start by computing iterates $F^n \eta_{\bar{v}}$. We translate them so that they all be centred at 0 (i.e. $J_{c_+}(T^{-j_n} F^n \eta_{\bar{v}}) = 0$ where $j_n = J_{c_+}(F^n \eta_{\bar{v}})$) and we look for a limit function. That is to say, we consider the sequence $\{T^{-j_n} F^n \eta_{\bar{v}}\}_{n \in \mathbb{N}}$.

Then we prove that $\liminf_{n \to \infty} (j_{n+m} - j_n) = m\bar{v}$. This property is employed together with an arithmetical lemma in order to ensure the existence of a strictly increasing sequence $\{n_k\}$ such that for all m we have

$$\lim_{k \to \infty} (j_{n_k+m} - j_{n_k}) = m\bar{v} \ .$$

By Helly's Selection Theorem[3] the sequence $\{T^{-j_{n_k}} F^{n_k} \eta_{\bar{v}}\}_{k \in \mathbb{N}}$ has a convergent subsequence which, without loss of generality, we assume to be the same sequence, i.e. there exists $\eta_\infty \in S_{\bar{v},c_+}$ such that

$$\eta_\infty = \lim_{k \to \infty} T^{-j_{n_k}} F^{n_k} \eta_{\bar{v}} \ .$$

[3] Helly's Selection Theorem states that if $\{f_n\}$ is a sequence of monotonically increasing functions on $\mathbb{R}$ with $0 \leq f_n(x) \leq 1$ for all x and n, then there is a function f and a sequence $\{n_k\}$ such that

$$f(x) = \lim_{k \to \infty} f_{n_k}(x) \quad \text{for every } x \in \mathbb{R} \ .$$

See Chap. 10 in [9] or exercise 13, Chap. 7 in [11].

Now we consider the sequence $T^{-m\bar{v}}F^m\eta_\infty$. Since $\eta_\infty \in S_{\bar{v},c_+}$, this sequence has the following property

$$\eta_{\bar{v}} \leq T^{-(m+1)\bar{v}}F^{m+1}\eta_\infty \leq T^{-m\bar{v}}F^m\eta_\infty \ .$$

Therefore it converges to a limit function

$$\phi = \lim_{m\to\infty} T^{-m\bar{v}}F^m\eta_\infty \ .$$

By continuity of F, the function $\phi \in \mathcal{I}$ satisfies the front equation $T^{\bar{v}}\phi = F\phi$ and is such that $\lim_{x\to+\infty}\phi(x) = 1$. However, one cannot conclude that $\lim_{x\to-\infty}\phi(x) = 0$ but only that

$$\lim_{x\to-\infty}\phi(x) \in \{0, c\} \ .$$

In order to complete the proof, one first shows that if f and h are such that

$$f'(c) = +\infty \text{ and } \inf\{x \in \mathbb{R} \ : \ h(x) > 0\} = -\infty$$

then $\lim_{x\to-\infty}\phi(x) = 0$ and the existence of fronts is proved in this case. In the general case, the conclusion follows

(1) by showing that any f (resp. h) can be approximated to arbitrary accuracy by local maps (resp. distribution functions) satisfying the previous assumptions and

(2) by showing that the limit of a sequence of extended bistable maps having fronts with converging velocities possesses itself a front (with the limit velocity).

3 Extended Circle Maps

3.1 Frenkel-Kontorova Models and Extended Circle Maps

One-dimensional chains of particles coupled by springs and placed in a periodic potential are represented by doubly infinite real sequences $\{u_s\}_{s\in\mathbb{Z}}$ (u_s represents the location of the s-th particle). In the dissipative limit, the dynamics of such chains, when driven by a constant force, are described by the gradient of a Frenkel-Kontorova (FK) functional (see the chapters by Floría, Baesens and Gómez-Gardeñes and by Baesens). It means that the sequences evolve according to the differential equation

$$\partial_t u_s = V'(u_s) + D + (u_{s-1} - 2u_s + u_{s+1})$$

where the potential V is periodic $V(x+1) = V(x)$ and $D \in \mathbb{R}$ is the driving force.

This equation is a special case of the following one

$$\partial_t u_s = -\left(g_2'(u_{s-1}, u_s) + g_1'(u_s, u_{s+1})\right) + D$$

where $g : \mathbb{R}^2 \to \mathbb{R}$ is a C^2 function satisfying the periodic condition

$$g(x+1, y+1) = g(x, y)$$

and such that the partial derivative $g_{12}''(x, y) \leq 0$ for all $(x, y) \in \mathbb{R}^2$ (twist condition).

This section concerns the dynamics of discrete time analogues of such equation; namely discrete time dynamical system defined by

$$u_s^{t+1} = u_s^t - \varepsilon \left(g_2'(u_{s-1}^t, u_s^t) + g_1'(u_s^t, u_{s+1}^t)\right) + \varepsilon D \tag{9}$$

where $t \in \mathbb{Z}$ and $\varepsilon > 0$ is the discretisation step. Special emphasis will be put on travelling wave solutions whose shape is an increasing periodic (in a suitable sense) function. Precisely, our concern will be with orbits given by

$$u_s^t = \psi\left(\alpha s + \nu_\alpha t\right) \tag{10}$$

where ψ is an increasing function such that $\psi\left(x+1\right) = \psi\left(x\right) + 1$. The number $\alpha > 0$ is called the mean spacing (wave number) of the wave and the number ν_α is called the rotation number (frequency).

A special case of travelling waves is when $\nu_\alpha = 0$ for which the configurations are stationary. In particular, according to relation (9), the existence of such stationary configuration for $D = 0$ is nothing else than the famous Aubry-Mather Theorem [8].

Just as done for bistable extended maps, we extend the analysis to systems with continuous space variable. That is to say, rather than considering sequences $u_s : \mathbb{Z} \to \mathbb{R}$, we consider functions $u(x) : \mathbb{R} \to \mathbb{R}$. In this larger phase space, the dynamics writes $u^{t+1} = Fu^t$ where

$$\left(Fu\right)\left(x\right) = u\left(x\right) - \varepsilon\left(g_2'(u\left(x-1\right), u\left(x\right)) + g_1'(u\left(x\right), u\left(x+1\right))\right) + \varepsilon D\,. \tag{11}$$

In order to deal with separate sets of travelling waves for distinct mean spacing, for each $\alpha > 0$, we consider the set (see Fig. 4)

$$\mathcal{N}_\alpha = \left\{u : u : \mathbb{R} \to \mathbb{R},\ u \text{ increasing and } u(x + \tfrac{1}{\alpha}) = u\left(x\right) + 1\right\}\,.$$

Since for any function ψ satisfying $\psi(x+1) = \psi(x) + 1$ the function $\phi\left(x\right) := \psi\left(\alpha x\right)$ belongs to $\mathcal{N}_\alpha$, the travelling wave solutions (10) can be written as follows

$$u^t = T^{-\frac{\nu_\alpha}{\alpha} t}\phi$$

where $\phi \in \mathcal{N}_\alpha$ and T^v is again the translation operator defined by $T^v u(x) = u(x - v)$.

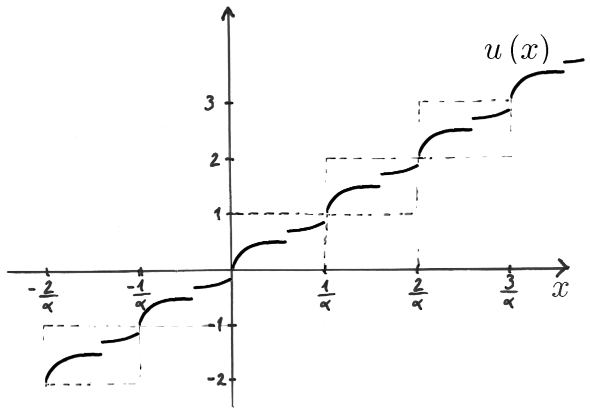

Fig. 4. A function u in $\mathcal{N}_\alpha$

Note that horizontal translations in $\mathcal{N}_\alpha$ can be viewed as vertical ones. So it is indifferent to view travelling waves either as propagating vertically or horizontally.

Just as bistable extended map do, the maps defined (11) commute with translations and are continuous with respect to pointwise convergence. In addition they can be shown to be increasing.

Lemma 3.1. [7] *For every $L > 0$, there exists $\varepsilon_L > 0$ such that, for every $\varepsilon \in (0, \varepsilon_L]$, $\alpha \in (0, L]$ and $u, v \in \mathcal{N}_\alpha$, we have $u \leq v$ implies $Fu \leq Fv$.*

At once, the map F satisfies the following properties

(a) For some $\alpha > 0$, or for all $\alpha > 0$, F maps $\mathcal{N}_\alpha$ into $\mathcal{N}_\alpha$.
(b) F is increasing, $u \leq v \Rightarrow Fu \leq Fv$.
(c) F is periodic, $F(u+1) = F(u) + 1$.
(d) F is homogeneous, $T^v F = F T^v \quad \forall v \in \mathbb{R}$.
(e) F is continuous, $\forall x \in \mathbb{R} \ \lim_{n\to\infty} u_n(x) = u(x) \Rightarrow \forall x \in \mathbb{R} \ \lim_{n\to\infty} Fu_n(x) = Fu(x)$.

The results in this section, in particular the uniqueness of rotation number (Proposition 3.1) and the existence of travelling waves (Theorem 3.1), hold for any map F which satisfies the properties a) to e). Such maps are called (spatially) **extended circle maps**.

The properties b), c) and d) imply that every set $\mathcal{N}_\alpha$ on which an extended circle map F is defined is invariant under the action of F. So the dynamics is well-defined.

The dynamics of functions with negative mean spacing (i.e. $\mathcal{N}_\alpha$ with $\alpha < 0$) is also included in this framework. Indeed, it suffices to apply the inversion $x \mapsto -x$ and to analyse the subsequent extended circle map. In the case $\alpha = 0$, the dynamics reduces to that of a lift of circle map (see below).

3.2 Coupled Lift of Circle Maps

A special example of extended circle map are the mappings which are formally identical to the bistable extended maps of the first section, namely

$$Fu = h * f \circ u \,. \tag{12}$$

However f is now a lift of a circle map (i.e. f is increasing, continuous map from $\mathbb{R}$ into itself for which $f(x+1) = f(x) + 1$ for all x) and h is now a distribution function satisfying $\int_{\mathbb{R}} |x| dh(x) < \infty$ [7]. Such maps cannot be interpreted as (extended) lift of coupled circle maps. Indeed for any integer function $n : \mathbb{R} \to \mathbb{N}$, we have $F(u+n) = F(u) + h * n$. But in general the function $h * n$ is not an integer function.

Anyway, coupled lift of circle maps (with discrete distribution function) can be interpreted as models of chains diffusively coupled particles in titled periodic potential (Frenkel-Kontorova models).

In the uncoupled case $h = H$ (where H is the Heaviside function) that is to say in the case where $Fu = f \circ u$, then Theorem 3.1 applied with $\alpha = 1$ states the existence of a semi-conjugacy to some translation for any lift of circle map f. Indeed, it states the existence of a lift of a circle map $\phi \in \mathcal{N}_1$ such that $f \circ \phi = T^{-\nu_1}\phi$ where ν_1 is the rotation number of F in $\mathcal{N}_1$ (see Proposition 3.1), the rotation number of f indeed.

3.3 Rotation Number of Extended Circle Maps

Unlike the analysis of fronts in bistable extended map for which the uniqueness of the velocity has been shown once a solution has been exhibited, the proof of existence of travelling waves with periodic shape begins with uniqueness of the rotation number.

In addition, the proof itself is simpler because the set $\mathcal{N}_\alpha$ containing the solutions are compact. This is also a reason why the proof extends to arbitrary circle maps F and holds not only for maps of the form $h * f \circ u$ (or their extension $\sum_k h_k * f_k \circ u$).

Horizontal displacements will be measured by using the reference zero:

$$J(u) := J_0(u) = \inf\{x \in \mathbb{R} \ : \ u(x) \geq 0\}$$

which is finite for every $u \in \mathcal{N}_\alpha$ with $\alpha > 0$ and satisfies the properties $J(T^\nu u) = J(u) + \nu$ and $u \leq v$ implies $J(u) \geq J(v)$.

The existence and the uniqueness of the rotation number for extended circle maps is given in the following statement.

Proposition 3.1. *Let F be an extended circle map defined on $\mathcal{N}_\alpha$ for some $\alpha > 0$. For every $u \in \mathcal{N}_\alpha$ and every $x \in \mathbb{R}$, the limit $\nu_\alpha := \lim_{t\to\infty} \frac{F^t u(x)}{t}$ exists and does not depend on x nor on u.*

Furthermore, we have $\left|J(F^t u) + \frac{\nu_\alpha t}{\alpha}\right| \leq \frac{2}{\alpha}$ *for all* $t \in \mathbb{N}$ *. Hence*

$$\nu_\alpha = \lim_{t\to\infty} \frac{F^t u(x)}{t} = -\,\alpha \lim_{t\to\infty} \frac{J(F^t u)}{t}\ .$$

The existence of the rotation number $\nu_\alpha = \lim_{t\to\infty} \frac{F^t u(x)}{t}$ extends to functions which need not be periodic but which can be sandwiched between two functions in $\mathcal{N}_\alpha$, see [7] for more details.

Proof: Every function $u \in \mathcal{N}_\alpha$ satisfies the inequalities

$$\varphi_\alpha^- \leq T^{-J(u)} u < \varphi_\alpha^+ \tag{13}$$

where $u < v$ means $u \leq v$ and $u \neq v$ and where the functions φ_α^- and φ_α^+ are defined by $\varphi_\alpha^-(x) = \lceil \alpha x \rceil - 1$ and $\varphi_\alpha^+ = \lfloor \alpha x \rfloor + 1$ for all $x \in \mathbb{R}$, see Fig. 5.

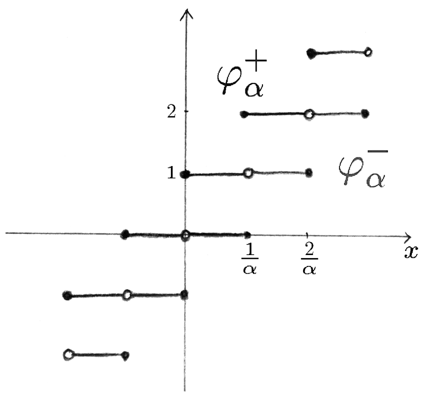

Fig. 5. The functions φ_α^- and φ_α^+

The quantity $j_t := J(F^t \varphi_\alpha^-)$ is finite for every $t \in \mathbb{N}$. The inequalities (13) imply $\varphi_\alpha^- \leq T^{-j_t} F^t \varphi_\alpha^-$ and $T^{-(j_t - \frac{1}{\alpha})} F^t \varphi_\alpha^+ < \varphi_\alpha^+$ because $J(F^t \varphi_\alpha^+) = j_t - \frac{1}{\alpha}$ for all t. Applying F^s, we obtain

$$F^s \varphi_\alpha^- \leq T^{-j_t} F^{t+s} \varphi_\alpha^- \quad \text{and} \quad T^{-(j_t - \frac{1}{\alpha})} F^{t+s} \varphi_\alpha^+ \leq F^s \varphi_\alpha^+$$

and then $j_s \geq j_{t+s} - j_t$ and $(j_{t+s} - \frac{1}{\alpha}) - (j_t - \frac{1}{\alpha}) \geq j_s - \frac{1}{\alpha}$. The sub-additivity of the sequence $\{j_t\}_{t\in\mathbb{N}}$ and the super-additivity of $\{j_t - \frac{1}{\alpha}\}_{t\in\mathbb{N}}$ imply that the following limit exists and is finite

$$\lim_{t\to\infty} \frac{j_t}{t} = \inf_{t>0} \frac{j_t}{t} = \sup_{t>0} \frac{j_t - \frac{1}{\alpha}}{t}\ .$$

We denote this quantity by $-\frac{\nu_\alpha}{\alpha}$. A consequence is the following important inequality

$$-\frac{\nu_\alpha}{\alpha}t \le j_t \le -\frac{\nu_\alpha}{\alpha}t + \frac{1}{\alpha}\,, \quad \forall t \in \mathbb{N}\,.$$

We are now about to prove the existence and uniqueness of the rotation number for the function φ_α^-. By applying the inequalities (13) to the function $F^t\varphi_\alpha^-$, we obtain

$$\varphi_\alpha^- - \lceil j_t\alpha\rceil = T^{\frac{1}{\alpha}\lceil j_t\alpha\rceil}\varphi_\alpha^- \le F^t\varphi_\alpha^- < T^{\frac{1}{\alpha}\lfloor j_t\alpha\rfloor}\varphi_\alpha^+ = \varphi_\alpha^+ - \lfloor j_t\alpha\rfloor\,.$$

As a consequence, for every $x \in \mathbb{R}$ we have

$$\nu_\alpha = -\lim_{t\to\infty}\tfrac{\lfloor j_t\alpha\rfloor}{t} \le \liminf_{t\to\infty}\tfrac{F^t\varphi_\alpha^-(x)}{t} \le \limsup_{t\to\infty}\tfrac{F^t\varphi_\alpha^-(x)}{t} \le -\lim_{t\to\infty}\frac{\lceil j_t\alpha\rceil}{t} = \nu_\alpha$$

the rotation number associated with φ_α^- exists and does not depend on x. In addition, $\varphi_\alpha^- + 1 \le \varphi_\alpha^+ \le \varphi_\alpha^- + 2$ and thus the rotation number also exists and does not depend on x for the function φ_α^+. Finally, by applying the inequalities (13) we conclude the same results for any $u \in \mathcal{N}_\alpha$ □

3.4 Existence of Travelling Waves

Proposition 3.1 claimed that every configuration in $\mathcal{N}_\alpha$ propagates asymptotically with a unique (horizontal) velocity. The main theorem below claims the existence of a configuration which the action of F amounts to a translation by $-\nu_\alpha/\alpha$.

Theorem 3.1. [7] *Let F be an extended circle map defined on $\mathcal{N}_\alpha$ for some $\alpha > 0$. There exists $\phi \in \mathcal{N}_\alpha$ such that $F\phi = T^{-\frac{\nu_\alpha}{\alpha}}\phi$.*

The proof is similar to the proof of front existence. As said before, the main difference resides in the compactness of $\mathcal{N}_\alpha$ which considerably simplifies the proof.

In a first step, we consider the set of sub-solutions of the travelling wave equation. Given $\nu \in \mathbb{R}$, we define

$$\mathcal{S}(\nu) = \left\{u \in \mathcal{N}_\alpha \;:\; T^{\frac{\nu}{\alpha}}Fu \le u \text{ and } J(u) = 0\right\}\,.$$

Next we show the rotation number can be defined using these sets:

Lemma 3.2. $\nu_\alpha = \inf\{\nu \in \mathbb{R} \;:\; \mathcal{S}(\nu) \ne \emptyset\}$.

Proof of the Lemma: Given $t \in \mathbb{N}$, let the function φ_t be defined by

$$\varphi_t(x) = \min_{0\le s<t}\left\{T^{-\frac{s}{t}(j_t-\frac{1}{\alpha})}F^s\varphi_\alpha^+(x)\right\}, \quad \forall x \in \mathbb{R}$$

where j_t and φ_α^+ were introduced in the proof of Proposition 3.1. The fact that $\varphi_\alpha^+ \in \mathcal{N}_\alpha$ and the properties of F ensure that $\varphi_t \in \mathcal{N}_\alpha$ for every t. Thus all $J(\varphi_t)$ are finite. Moreover, by monotony of F, we have $T^{-\frac{1}{t}(j_t-\frac{1}{\alpha})}F\varphi_t \le T^{-\frac{s+1}{t}(j_t-\frac{1}{\alpha})}F^{s+1}\varphi_\alpha^+$ for every $0 \le s < t$. This implies that

$$T^{-\frac{1}{t}(j_t-\frac{1}{\alpha})}F\varphi_t \leq \min_{1\leq s\leq t} T^{-\frac{s}{t}(j_t-\frac{1}{\alpha})}F^s\varphi_\alpha^+ \leq \min_{0\leq s<t} T^{-\frac{s}{t}(j_t-\frac{1}{\alpha})}F^s\varphi_\alpha^+ = \varphi_t$$

because $T^{-(j_t-\frac{1}{\alpha})}F^t\varphi_\alpha^+ \leq \varphi_\alpha^+$ as indicated by the right inequality (13). We have shown that the set $\mathcal{S}(-\frac{\alpha}{t}(j_t-\frac{1}{\alpha}))$ is not empty for every $t>0$. Therefore, we have

$$\nu_\alpha = -\alpha \lim_{t\to\infty} \frac{1}{t}\left(j_t - \frac{1}{\alpha}\right) \geq \inf\left\{\nu\in\mathbb{R} \ : \ \mathcal{S}(\nu)\neq\emptyset\right\}.$$

On the other hand, we assume that $u\in\mathcal{S}(\nu)\neq\emptyset$ for some $\nu\in\mathbb{R}$. Then $\varphi_\alpha^- \leq u$ by relation (13) and thus $F^t\varphi_\alpha^- \leq F^tu \leq T^{-\frac{\nu}{\alpha}t}u$ which implies $j_t \geq -\frac{\nu}{\alpha}t$, i.e. $\nu \geq -\alpha\frac{j_t}{t}$ for all $t>0$. Consequently, we have

$$\inf\left\{\nu\in\mathbb{R} \ : \ \mathcal{S}(\nu)\neq\emptyset\right\} \geq \nu_\alpha .$$

□

As in the proof of front existence, the second step consists of considering a minimal sub-solutions, namely we consider the function

$$\eta_\nu(x) = \inf_{u\in\mathcal{S}(\nu)} u(x), \quad \forall x\in\mathbb{R} .$$

It turns out that $\mathcal{S}(\nu_\alpha)\neq\emptyset$ and $\eta_{\nu_\alpha}\in\mathcal{S}(\nu_\alpha)$.

In a third step, we consider the sequence $\left\{T^{n\frac{\nu_\alpha}{\alpha}}F^n\eta_{\nu_\alpha}\right\}_{n\in\mathbb{N}}$. By monotony and homogeneity, we have

$$T^{(n+1)\frac{\nu_\alpha}{\alpha}}F^{n+1}\eta_{\nu_\alpha} \leq T^{n\frac{\nu_\alpha}{\alpha}}F^n\eta_{\nu_\alpha}, \quad \forall n\in\mathbb{N} .$$

In addition, one can show that $T^{\frac{1}{\alpha}}\varphi_\alpha^- \leq T^{n\frac{\nu_\alpha}{\alpha}}F^n\eta_{\nu_\alpha}$ and hence that the sequence is bounded from below. Consequently, this sequence converges pointwise to the limit function $\phi\in\mathcal{N}_\alpha$ which satisfies $F\phi = T^{-\nu_\alpha/\alpha}\phi$. We refer to [7] for more details.

3.5 Continuity of the Rotation Number

Just as the front velocity, the rotation number ν_α varies continuously with changes of extended circle maps (and in particular with their parameters).

For coupled lift of circle maps $Fu = h * f\circ u$, changes are the same as before; namely pointwise convergence for the local map f and convergence in the Hausdorff topology for the distribution function h (an additional condition is needed to ensure that the distribution functions h satisfy $\int_{\mathbb{R}}|x|\,dh(x)<\infty$, see Lemma 3.3 in [7]).

For the discrete time version (9) of the Frenkel-Kontorova model, a continuous dependence with parameters and with the generating function has been shown [7]. All these results are deduced from the following statement valid for arbitrary extended circle maps. If $\lim_{n\to\infty}\sup_{u\in\mathcal{N}_\alpha} d(F_n(u),F(u))=0$ where the distance $d(\cdot,\cdot)$ is the Hausdorff distance, then $\lim_{n\to\infty}\nu_\alpha(F_n)=\nu_\alpha(F)$.

In complement to continuity with respect to changes in the map, the rotation number depends continuously on the mean spacing α. This is proved by using bigger spaces $\mathcal{M}_{\alpha',\alpha''}$ which contain $\mathcal{N}_\alpha$ for every $\alpha'\leq\alpha\leq\alpha''$.

3.6 Extended Circle Maps with Vanishing Rotation Number

Recall that Aubry-Mather Theorem states that any Frenkel-Kontorova functional possess stationary configurations for every mean spacing α [8].

In the present framework, this amounts to say that the map F defined by (11) with $D = 0$ has, for every α, a fixed point in $\mathcal{N}_\alpha$. By Proposition 3.1 and Theorem 3.1, this result would follow from the fact that $\nu_\alpha = 0$ for all α.

It turns out that the property $\nu_\alpha = 0$ for all $\alpha > 0$ is not limited to the model (11) with $D = 0$. As stated in the next statement, it extends to any extended circle map satisfying some symmetry condition.

Theorem 3.2. *If there exists a lift of a circle map $\tilde{f}$ such that the following relation holds*

$$\int_0^{\frac{1}{\alpha}} (Fu - u) d(\tilde{f} \circ u) = 0 \tag{14}$$

for every continuous function $u \in \mathcal{N}_\alpha$ ($\alpha > 0$), then the rotation number $\nu_\alpha = 0$.

The proof essentially relies on various properties of the Lebesgue-Stieltjes integral [7]. The fact that the model (11) with $D = 0$ satisfies this property, however, is elementary and claim in our final statement.

Proposition 3.2. *For every generating function g, every $\alpha > 0$, and every $\varepsilon > 0$, the map*

$$F_\varepsilon u(x) = u(x) - \varepsilon \left(g_2'(u(x-1), u(x)) + g_1'(u(x), u(x+1)) \right) \quad \forall x \in \mathbb{R}\,,$$

satisfies the condition (14) with $\tilde{f}(x) = x$.

Indeed by using $T^{-\frac{1}{\alpha}} u = u + 1$, we have

$$\begin{aligned}
&\int_0^{\frac{1}{\alpha}} (F_\varepsilon u - u) du \\
&\quad = -\varepsilon \int_0^{\frac{1}{\alpha}} g_2'(u(x), u(x+1)) du(x+1) - \varepsilon \int_0^{\frac{1}{\alpha}} g_1'(u(x), u(x+1)) du(x) \\
&\quad = -\varepsilon \int_0^{\frac{1}{\alpha}} dg(u(x), u(x+1)) = 0
\end{aligned}$$

for every continuous function $u \in \mathcal{N}_\alpha$.

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