Erratum of the paper

Topological properties of linearly coupled expanding maps lattices

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1 Proof of Proposition 3.2

The proof of Proposition 3.2 is not correct because there are sets in $\ell^\infty(\mathbb{Z})$ for which every one-dimensional canonical projection contains an interval, say $I_i$, but which do not cover the product of intervals $\bigotimes_{s\in\mathbb{Z}} I_s$. The image of $[0,1]^\mathbb{Z}$ by a convolution satisfying (H2) and (H3) is an example of such a set. This section is a proof, which we believe to be correct.

**Proof:** For each $i\in\{1,\cdots,N\}$, let $u^{(i)}$ be the middle point of $f(I_i)$ and $\delta_i = \frac{|f(I_i)|}{2}$, i.e.

\[ f(I_i) = u^{(i)} + [-\delta_i, \delta_i]. \]

For each $i$, the condition $I_j \subset \text{Int} f(I_i)$ implies the existence of $0 < \alpha_{i,j} < 1$ such that

\[ I_j \subset u^{(i)} + [-\alpha_{i,j} \delta_i, \alpha_{i,j} \delta_i]. \]

In other words, there exists $0 < \alpha < 1$ such that for any $\omega, \omega' \in \{1,\cdots,N\}^\mathbb{Z}$ so that $I_{\omega'} \subset \text{Int} f(I_{\omega}), s \in \mathbb{Z}$, we have

\[ I_{\omega'} \subset u^{\omega} + \bigotimes_{s\in\mathbb{Z}} [-\alpha \delta_{\omega_s}, \alpha \delta_{\omega_s}] \subset F(I_{\omega}), \]

where $u^{\omega} = u^{(\omega_s)}$ for every $s$. The left inclusion shows that one only has to show that if $||I_d - L||$ is sufficiently small, we have

\[ u^{\omega} + \bigotimes_{s\in\mathbb{Z}} [-\alpha \delta_{\omega_s}, \alpha \delta_{\omega_s}] \subset L \circ F(I_{\omega}). \]

The latter is a consequence of the following result.

**Lemma 1.1** For any $\gamma > 1$, there exists $\varepsilon_\gamma > 0$ such that for any coupling satisfying $||I_d - L|| < \varepsilon_\gamma$, $L^{-1}$ exists and for arbitrary $\omega = \{\omega_s\}_{s\in\mathbb{Z}} \in \{1,\cdots,N\}^\mathbb{Z}$, we have

\[ L^{-1}(\bigotimes_{s\in\mathbb{Z}} [\delta_{\omega_s}, \delta_{\omega_s}]) \subset \bigotimes_{s\in\mathbb{Z}} [-\gamma \delta_{\omega_s}, \gamma \delta_{\omega_s}]. \]
Indeed, if \( \|\text{Id} - L\| < \varepsilon_\gamma \), then linearity implies that
\[
L^{-1}(\bigotimes_{s \in \mathbb{Z}} [\alpha \delta_{\omega_s}, \alpha \delta_{\omega_s}]) \subset \bigotimes_{s \in \mathbb{Z}} [-\alpha \gamma \delta_{\omega_s}, \alpha \gamma \delta_{\omega_s}].
\]
Let \( 1 < \gamma < \frac{1}{2} \) and choose \( \varepsilon_\gamma \) smaller if necessary so that, in addition to the previous relation, the condition \( \|\text{Id} - L\| < \varepsilon_\gamma \) implies
\[
\|(L^{-1} u^\omega)_s - u^\omega_s\| < (1 - \alpha \gamma) \delta_{\omega_s}, \quad s \in \mathbb{Z}.
\]
Consequently, for any \( L \) such that \( \|\text{Id} - L\| < \varepsilon_\gamma \), we have
\[
L^{-1}(u^\omega + \bigotimes_{s \in \mathbb{Z}} [-\alpha \delta_{\omega_s}, \alpha \delta_{\omega_s}]) \subset u^\omega + \bigotimes_{s \in \mathbb{Z}} [\delta_{\omega_s}, \delta_{\omega_s}] = F(I_\omega),
\]
from which the desired result follows by applying \( L \).

**Proof of the Lemma:** In all the proof, we assume that \( \varepsilon_\gamma < 1 \). As a consequence, \( L^{-1} \) exists and is a convolution. Let \( \{\ell_n\}_{n \in \mathbb{Z}} \) be the sequence representing \( L^{-1} \). Assume that \( \ell_0^{-1} > 0 \) and let \( \delta = \max_{1 \leq i \leq N} \delta_i \).

Take any \( \omega \in \{1, \cdots, N\}^\mathbb{Z} \) and any \( u \in \bigotimes_{s \in \mathbb{Z}} [\delta_{\omega_s}, \delta_{\omega_s}] \). We have
\[
(L^{-1} u)_s \leq \ell_0^{-1} \delta_{\omega_s} + \delta \sum_{n \neq 0} |\ell_n^{-1}|, \quad s \in \mathbb{Z}.
\]
We use this bound to show that
\[
(L^{-1} u)_s \leq \gamma \delta_{\omega_s}, \quad s \in \mathbb{Z}, \tag{1}
\]
where \( \gamma > 1 \) is given and provided that \( \|\text{Id} - L\| \) is sufficiently small.

Given \( \gamma > 1 \) and \( i \in \{1, \cdots, N\} \), there exists \( \eta_i > 0 \) such that
\[
(1 + \eta_i) \delta_i + \delta \eta_i \leq \gamma \delta_i.
\]
Now, we have
\[
\ell_0^{-1} \leq \sum_{n \in \mathbb{Z}} |\ell_n^{-1}| = \|L^{-1}\| \leq \sum_{k \in \mathbb{Z}^+} \|\text{Id} - L\|^k \leq \frac{1}{1 - \varepsilon},
\]
whenever \( \|\text{Id} - L\| < \varepsilon \). Consequently, there exists \( \varepsilon_2 > 0 \) such that \( \|\text{Id} - L\| < \varepsilon_2 \) implies that
\[
\ell_0^{-1} \leq \min_{1 \leq i \leq N} 1 + \eta_i
\]
Moreover, estimating the sequence \( \{\ell_n^{-1}\} \) using the Neumann series defining \( L^{-1} \), we obtain
\[
\ell_0^{-1} \geq 1 - \sum_{k \in \mathbb{N}} \|\text{Id} - L\|^k \geq \frac{1 - 2\varepsilon}{1 - \varepsilon},
\]
whenever \( \|\text{Id} - L\| < \varepsilon \). (This shows that \( \ell_0^{-1} > 0 \) when \( \varepsilon < \frac{1}{2} \)). Consequently, there exists \( \varepsilon_3 > 0 \) such that \( \|\text{Id} - L\| < \varepsilon_3 \) implies that
\[
\sum_{n \neq 0} |\ell_n^{-1}| = \|L^{-1}\| - \ell_0^{-1} \leq \frac{2\varepsilon_3}{1 - \varepsilon_3} \leq \min_{1 \leq i \leq N} \eta_i.
\]
We conclude that the inequality (1) holds when \( \|\text{Id} - L\| < \varepsilon_\alpha = \min\{1, \frac{1}{2}, \varepsilon_2, \varepsilon_3\} \).

Using linearity, we obtain corresponding lower bound for \( (L^{-1} u)_s \) and the Lemma follows. \( \square \)
2 Proof of Proposition 3.3

In the proof of Proposition 3.3, the intersections
\[ \mathcal{F}_{f, L}^{-t}(L_{\omega^t+1}) \cap \mathcal{F}_{f, L}^{-t+1}(L_{\omega^t}) \neq \emptyset, \]
do not suffice to ensure that \( J_{\omega^1, \ldots, \omega^t} = \bigcap_{k=1}^{t} \mathcal{F}_{f, L}^{-k+1}(L_{\omega^k}) \) is non-empty. However, a classical argument shows that when the following inclusions hold (which is the case by Proposition 3.2)
\[ L_{\omega^{t+1}} \subset \mathcal{F}_{f, L}(L_{\omega^t}), \quad t \in \mathbb{N}, \]
every admissible cylinder \( J_{\omega^1, \ldots, \omega^t} \) is non-empty.