Comments about "Population Dynamics in Heterogeneous Environments: a Discrete Model"

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Abstract

In [1], a space-time discrete model for population dynamics on heterogeneous landscapes has been introduced and studied. Here we prove that in the simple case of a homogeneous fitness, the asymptotic population is homogeneous.

We briefly recall the definitions. Given $L \in \mathbb{N}$, let $f = \{f_s\}_{1\leqslant s\leqslant L}$ with $f_s \geqslant 0$ be the fitness configuration and let

$$m = \min_s f_s \quad \text{and} \quad M = \max_s f_s.$$

We assume that $M \leqslant 4$. The phase space is the following simplex

$$\mathcal{M}_M = \left\{ u \in [0,1]^L \ : \ \sum_{s=1}^{L} u_s \leqslant \frac{M}{4} \right\}.$$

In this set, the dynamics is generated by the map T acting on configurations $u = \{u_s\}_{s=1}^{L}$ and defined by

$$(Tu)_s = f_s u_s (1 - \sum_{r=1}^{L} u_r) + D(u_{s-1} - 2u_s + u_{s+1}), \quad 1 \leqslant s \leqslant L,$$

with periodic boundary conditions (i.e. $u_0 = u_L$ and $u_{L+1} = u_1$) and where $0 \leqslant D \leqslant m\frac{4-M}{8}$. The conditions on M and on D imply that $T(\mathcal{M}_M) \subset \mathcal{M}_M$ and the dynamics is meaningful [1].

To analyse the asymptotic behaviour, it is convenient to introduce the Fourier transform. Let $(\mathcal{F}u)_k = \sum_{s=1}^{L} u_s e^{(\frac{2i\pi ks}{L})}$, $0 \leqslant k < L$ be the Fourier transform of a configuration in $\mathbb{R}^L$. Let $U^t = \mathcal{F}u^t$ and $F = \mathcal{F}f$ be respectively the Fourier transforms of an orbit's component and of the fitness. We have

$$U_k^{t+1} = \frac{1 - U_0^t}{L} \sum_{j=0}^{L-1} F_j U_{k-j \bmod L}^t + \lambda_k D U_k^t, \quad 0 \leqslant k < L, \tag{1}$$

where $\lambda_k = -4\sin^2\left(\frac{\pi k}{L}\right)$.

In [1], it has been shown that if the fitness is homogeneous (i.e. if f_s does not depend on s), then any initial condition with mean value U_0^0 equals to a component of a periodic orbit of the logistic map, or in the basin of attraction of such an orbit, the asymptotic configuration is homogeneous. In other words, these initial configurations synchronize in time.

Here, we prove that this result extends to any initial condition. In short terms, if the fitness is homogeneous, so is the asymptotic population and we have synchronization.

Theorem 1 *If $f_s = \mu$ for all s and $0 < D < \mu\frac{4-\mu}{8}$ then for any $u \in \mathcal{M}_\mu$, we have*

$$\lim_{t\to\infty} \max_{0<k<L} |U_k^t| = 0.$$

It is interesting to compare this result to those in other models such as Coupled Map Lattices (CML). In the latter, synchronisation may occur only for strong coupling, i.e. when all the Fourier modes become contracting. This main difference in the behaviour becomes from the difference in the map defining the dynamics. Whereas in the present case the dynamics of the modes is linear because of the mean-field type of nonlinearity, in CML, the same dynamics is nonlinear because the original nonlinearity is local.

Proof: Let $\{U_0^t\}_{t\in\mathbb{N}}$ be an orbit of the logisitic map. As argued in [1], we may assume that $U_0^t > 0,\ t \in \mathbb{N}$. Otherwise, since $u \in \mathcal{M}_\mu$, the condition $U_0^{t_0} = 0$ implies that $U_k^t = 0,\ 0 \leqslant k < L,\ t \geqslant t_0$ and the result holds.

We define the following times, denoted t_n, by iterations. Given $n \in \mathbb{N}$, let $t_1 = 1$ and let

$$t_n = \inf\{j > t_{n-1}\ :\ U_0^j > U_0^{t_{n-1}}\},$$

if $n > 1$. It may happen that $t_n = \infty$ for some $n \in \mathbb{N}$. In this case, there is no t_k with $k > n$.

Using (1), one obtains that for any $t_n < t \leqslant t_{n+1}$, $n \in \mathbb{N}$, the dynamics of the positive Fourier modes factorizes as

$$|U_k^t| = \frac{U_0^t}{U_0^{t_n}} \prod_{j=0}^{t-t_n-1} \left| 1 + \lambda_k D \frac{U_0^{t_n+j}}{U_0^{t_n+j+1}} \right| |U_k^{t_n}|, \quad 0 < k < L.$$

By assumption on D, there exists $0 < \delta < 1$, such that $D \leqslant (1 - \frac{\delta}{2})\mu\frac{4-\mu}{8}$. Since $u \in \mathcal{M}_\mu$, we have $U_0^t \leqslant \frac{\mu}{4}$, $t \in \mathbb{N}$. Together with the definition of λ_k, this implies that for any n and any j, we have

$$-(1-\delta) \leqslant 1 - 4D\frac{1}{\mu(1 - U_0^{t_n+j})} = 1 - 4D\frac{U_0^{t_n+j}}{U_0^{t_n+j+1}} \leqslant 1 + \lambda_k D \frac{U_0^{t_n+j}}{U_0^{t_n+j+1}}.$$

On the other hand, since $\frac{U_0^{t_n+j}}{U_0^{t_n+j+1}} \geqslant \frac{1}{\mu}$, $D > 0$ and $\lambda_k < 0$, we have

$$1 + \lambda_k D \frac{U_0^{t_n+j}}{U_0^{t_n+j+1}} \leqslant 1 + \frac{D}{\mu} \max_k \lambda_k < 1 \quad 0 < k < L.$$

We deduce that for any $t_n < t \leqslant t_{n+1}$, we have

$$|U_k^t| \leqslant \frac{U_0^t}{U_0^{t_n}} \left(1 - \min\{\delta, -\frac{D}{\mu} \max_k \lambda_k\} \right)^{t-t_n} |U_k^{t_n}|, \quad 0 < k < L,$$

and $0 < 1 - \min\{\delta, -\frac{D}{\mu}\max_k \lambda_k\} < 1$. The results then follows by taking the limit $t \to \infty$ if $t_{n+1} = +\infty$ for some $n \in \mathbb{N}$ because of uniform boundedness of U_0^t.

If t_n is finite for all n, then we have

$$|U_k^t| \leqslant |U_k^{t_n}|, \quad t_n < t < t_{n+1}. \tag{2}$$

Moreover since U_0^t is restricted to $[0, \frac{\mu}{4}]$, for any $1 - \min\{\delta, -\frac{D}{\mu}\max_k \lambda_k\} < \eta < 1$, there exists $n' \in \mathbb{N}$ such that

$$\frac{U_0^{t_{n+1}}}{U_0^{t_n}} \leqslant \frac{\eta}{1 - \min\{\delta, -\frac{D}{\mu}\max_k \lambda_k\}}, \quad n > n'.$$

Therefore

$$|U_k^{t_{n+1}}| \leqslant \eta |U_k^{t_n}|, \quad 0 < k < L, \ n > n',$$

from which the Theorem follows when iterating and using (2). □

Erratum

– Page 1996, the assumption (H) has to be corrected in the following way:

(H) The fitness is homogeneous, $f_s = \mu$ for all s and the diffusion coefficient satisfies the inequalities $0 < D < \frac{\mu(4-\mu)}{8}$.

– Page 1997, the following comment is added line 4:

The map $x \to \mu x(1-x)$ has a periodic orbit for any value of μ. We denote by $\{x^t\}_{t\in\mathbb{N}}$ with $x^{t+\tau} = x^t$ for some $\tau \in \mathbb{N}$ such an orbit.

– Page 1997, in Statement (ii) of Corollary 4.3, the word "sum" should be "mean".

References

[1] B. Fernandez and V. Tereshko, *Population Dynamics in Heterogeneous Environments: A Discrete Model*, Int. J. Bif. & Chaos **10** (2000) 1993-2000.